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Aspects of $N_T \geq 2$ Topological Gauge Theories and D-Branes

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We comment on various aspects of topological gauge theories possessing $N_T \geq 2$ topological symmetry:

1. We show that the construction of Vafa-Witten and Dijkgraaf-Moore of 'balanced' topological field theories is equivalent to an earlier construction in terms of $N_T = 2$ superfields inspired by supersymmetric quantum mechanics.
2. We explain the relation between topological field theories calculating signed and unsigned sums of Euler numbers of moduli spaces.
3. We show that the topological twist of $N = 4$ $d = 4$ Yang-Mills theory recently constructed by Marcus is formally a deformation of four-dimensional super-BF theory.
4. We construct a novel $N_T = 2$ topological twist of $N = 4$ $d = 3$ Yang-Mills theory, a 'mirror' of the Casson invariant model, with certain unusual features (e.g. no bosonic scalar field and hence no underlying equivariant cohomology)
5. We give a complete classification of the topological twists of $N = 8$ $d = 3$ Yang-Mills theory and show that they are realised as world-volume theories of Dirichlet two-brane instantons wrapping supersymmetric three-cycles of Calabi-Yau three-folds and G_2-holonomy Joyce manifolds.
6. We describe the topological gauge theories associated to D-string instantons on holomorphic curves in K3s and Calabi-Yau 3-folds.

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1 Introduction

Recently, topological field theories with extended $N_T > 1$ topological symmetries have appeared in various contexts, e.g. in the discussion of S-duality in supersymmetric gauge theories [1, 2], as world volume theories of Dirichlet p-branes in string theory [3], and in a general discussion of 'balanced' or

critical topological theories [4]. Here we will comment on, explain, or expand on various aspects of these theories, thus complementing the already existing discussions of such models in the literature.

Let us begin by recalling that, in contrast to cohomological topological field theories with an extended (in particular $N_T > 2$) topological symmetry, $N_T = 1$ topological theories (the most prominent example being Donaldson-Witten or topological Yang-Mills theory [5]) are reasonably well understood (see e.g. [6] for a review). They typically capture the deformation complex of some underlying moduli problem, their partition function is generically zero because of fermionic zero modes while correlation functions correspond to intersection pairings on the moduli space.

In [7, 8] we investigated in some detail topological gauge theories of a particular kind possessing an $N_T = 2$ topological symmetry. These theories typically capture the de Rham complex and Riemannian geometry of some underlying moduli space, and they have the characteristic property of being 'critical', i.e. of possessing a generically non-zero partition function equalling the Euler number of the moduli space. These properties are obviously reminiscent of supersymmetric quantum mechanics and in [7, 8] we made this connection precise using the Mathai-Quillen formalism [9, 10, 11] and an $N_T = 2$ superfield version of a new variant of supersymmetric quantum mechanics based on the Gauss-Codazzi equations.

More recently there has been renewed interest in topological field theories possessing extended $N_T = 2$ topological symmetries. For example, in [1] Vafa and Witten considered a particular topologically twisted version of $N = 4$ $d = 4$ super-Yang-Mills theory to perform a strong coupling test of S-duality of the underlying supersymmetric theory. By analogy with the nomenclature for supersymmetric sigma models [12], we shall (as suggested in [13]) refer to this model as the A-twist of $N = 4$ $d = 4$ Yang-Mills theory or simply as the A-model. The partition function of this topological gauge theory equals the Euler number of instanton moduli space or, more precisely, in the case that the instanton moduli space is not connected, the sum of the seperate Euler numbers disregarding the relative orientations.

Along similar lines, in [14] a topological string theory calculating the Euler number of the (Hurwitz) moduli space of branched covers was constructed and shown to reproduce the large N expansion of two-dimensional Yang-Mills theory obtained by Gross and Taylor [15].

The constructions of [1] and [14] have recently been generalized by Dijkgraaf

and Moore [4] who christened these theories 'balanced' topological field theories and analyzed in detail the underlying $N_T = 2$ equivariant cohomology.

On the face of it the constructions of $N_T = 2$ theories given in [4] and [7, 8] respectively appear to be quite different. One purpose of this paper is to show that they are actually completely equivalent. This observation sheds light on both approaches as on the one hand it provides a cohomological underpinning to the construction of [7, 8] while on the other hand it gives an *a priori* explanation in terms of Riemannian geometry for why balanced topological field theories calculate Euler numbers.

We also pause to explain the relation between theories calculating Euler numbers with and without relative signs, as this turns out to be particularly transparent from the supersymmetric quantum mechanics point of view and is something we should have stated more clearly in [7].

A somewhat different kind of $N_T = 2$ symmetry, not falling into the above scheme of things, appears in 'the other' topological twist of $N = 4$ Yang-Mills theory (the B-model), mentioned in [16, 1] and recently constructed explicitly by Marcus [13]. We will show that the B-model is formally a deformation of four-dimensional super-BF theory [17, 6], i.e. the four-dimensional cohomological field theory (formally) describing moduli spaces of real flat connections.[1] The deformation in question deforms the cotangent bundle of the moduli space of real flat connections to the moduli space of complex flat connections and one expects that this does not change the topological aspects of the theory. One can then understand the $N_T = 2$ topological symmetry as a consequence of the complex nature of the moduli space at this particular point in the deformation space of super-BF theory.

This is reminiscent of the extended topological symmetries that can arise when $N_T = 1$ theories are formulated on manifolds with reduced holonomy groups, as in Donaldson-Witten theory on a Kähler manifold. In both cases this leads to a (Dolbeault) refinement of the original (de Rham) intersection theory model but does not change the theory in other respects.

Continuing on our stroll through the zoo of $N_T = 2$ theories, we construct a novel topological gauge theory in three dimensions by twisting the $N = 4$ $d = 3$ theory, the dimensional reduction of $N = 1$ $d = 6$ or $N = 2$ $d = 4$ theory, by the internal Lorentz group $SU(2)$. As a consequence, the potential three scalars (the internal components of the gauge field) transform as a vector V

[1] A slightly different description of this theory has been obtained by Labastida and Lozano (J.M.F. Labastida, private communication).

and this model has no bosonic scalars in its spectrum. Hence, as there is in particular no scalar ghost for ghost, the topological symmetry of this model cannot possibly correspond to equivariant cohomology and indeed we find that here the topological symmetries are strictly (and not only equivariantly, as in all the models above) nilpotent and anti-commuting. Another novel feature of this model is that while it is a cohomological gauge theory it permits bosonic Wilson loops (of $A - iV$) as fully topological observables.

Along rather different lines, Bershadsky, Sadov and Vafa [3] have recently made the beautiful observation that topologically twisted gauge theories appear completely naturally as (low-energy effective) world-volume theories of Dirichlet p-brane instantons in string theory (for recent reviews of these matters see [18, 19]). In particular, they showed that the three (partial) topological twists of $N = 4$ $d = 4$ Yang-Mills theory [16, 1, 13] are realized in this way.

Motivated by this, we classify the topological and partial topological twists of $N=8$ $d=3$ (the dimensional reduction of $N=4$ $d=4$) Yang-Mills theory. Once again one finds that all of them are naturally realized as world-volume theories of Dirichlet two-brane instantons, on special Lagrangian submanifolds of Calabi-Yau three-folds and associative submanifolds of G_2-holonomy seven-manifolds respectively. In the same spirit, we also describe the topological gauge theories associated to D-string instantons wrapping holomorphic curves in K3s and Calabi-Yau three-folds respectively.

We also expand slightly on the discussion given in [3] by showing that in all these theories the rotation group of the uncompactified dimensions (in the string interpretation) is realized as a global symmetry group of the world-volume theories - as required by the geometrical interpretation. Conversely, demanding such a symmetry determines the relevant class of branchings of the R-symmetry group and thus provides a short-cut to the construction of the desired topological theory via twisting.

Finally, we want to point out that to a certain extent similar constructions are possible in $d = 5$ and $d = 6$. For example, there should be a close relation between twisted theories on $M_4 \times T^2$, the (4+2)-dimensional generalization of the (4+1)-dimensional quantum mechanics model of [7, 8] and section 2, and the considerations in [20]. It is also straightforward to construct a topological gauge theory in $d = 5$. It appears to be related to the constructions and considerations in [21]. These issues are currently under investigation.

This paper is organized as follows. In section 2 we recall the supersymmetric

quantum mechanics based construction of topological field theories [7, 8, 11] and establish the link with the approaches of [1, 14, 4]. In section 3, we present a detailed comparison of the topological superfield approach of [7] with the approach of Dijkgraaf and Moore [4] based on $N_T \geq 1$ extended equivariant cohomology, establishing their equivalence. In section 4 we discuss a simplified quantization of $d = 4$ super-BF theory (section 4.1), show that it can be deformed to the action given in [13] (section 4.2), and introduce a novel topological twist of $N = 4$ $d = 3$ Yang-Mills theory. In section 5, we classify the topological twists of $N = 8$ $d = 3$ Yang-Mills theory (section 5.1), show that they are realized as world-volume theories of wrapped two-brane instantons on supersymmetric three-cycles (section 5.2), and extend the discussion to topological gauge theories associated with D-string instantons wrapping holomorphic curves (section 5.3).

2 $N_T = 2$ Topological Field Theories and Euler Numbers

It is well known that supersymmetric quantum mechanics [22] (SQM) can be regarded as the archetype of a cohomological field theory (for a review of SQM from this point of view see [6]). In particular, what is commonly known as the $N = 1/2$ model, calculating the index of the Dirac operator, generalizes to cohomological field theory with an $N_T = 1$ topological supersymmetry[2] describing the geometry of the Atiyah-Singer universal bundle and intersection theory on moduli spaces.

The partition function of the $N = 1$ SQM model, on the other hand, equals the Euler number of the target space, and it is thus natural to suspect that this generalizes to $N_T = 2$ topological field theories calculating the Euler number of some moduli space. How this can be accomplished was explained in detail in [7, 8] where the structure of topological gauge theories with the desired properties was examined from several different points of view. In particular, it was shown that actions for $N_T = 2$ superfields encode the Riemannian geometry of the field space, leading via localization and the Gauss-Bonnet theorem to the Euler number of a prescribed moduli space. We will briefly review this construction below.

Recently, an alternative approach to this kind of topological field theories, based on $N_T = 2$ equivariant cohomology, was presented in [4], generalizing

[2] In topological field theories it is more natural to count real scalar supercharges instead of spinors

the procedure employed in [1, 14]. By comparing the field content and action of the resulting theories with those obtained in [7, 8] (and their - straightforward - adaptation to sigma models), it is readily seen that the two constructions are equivalent. However, it is instructive to also explain this in terms of the relation between the (seemingly rather different) superfields appearing in these two approaches, and we shall do so in detail in section 3.

One of the properties the SQM-inspired approach [7, 8] makes particularly transparent is the relation between theories counting Euler numbers (or solutions of equations) with or without signs (recall that constructing topological field theories accomplishing the latter was the main motivation for the 'cofield' construction in [1, 14, 4]). This will be explained in section 2.4.

2.1 Review: Variants of Supersymmetric Quantum Mechanics

The principal aim of [7, 8] was to explain a construction of topological gauge theories formally calculating the Euler character of a specified moduli space $\mathcal{M}$ of connections. In its most pragmatic incarnation, this construction is expressed in terms of $N_T = 2$ superfields and an action $S_{\mathcal{M}}$ which consists of a kinetic term and a supersymmetric delta function constraint onto the desired moduli space. The relation between this approach and that by Vafa and Witten [1] and Dijkgraaf and Moore [4] based on equivariant cohomology will be explained in the next section.

To set the stage for this, however, in the following we will briefly recall the considerations based on supersymmetric quantum mechanics [8] which inspired the superfield formulation as they provide an *a priori* explanation for why the topological gauge theories constructed in this manner do what they are supposed to do. These remarks should also make it clear that the construction can be straightforwardly adapted to e.g. topological sigma models or topological gravity.

Let us start by recalling that the partition function of $N = 1$ ($N_T = 2$) supersymmetric quantum mechanics with target space a Riemannian manifold (X, g), whose action is schematically of the form

$$S_X = \int dt \; g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \text{superpartners} \quad , \tag{2.1}$$

equals the Euler nuber of X,

$$Z(S_X) = \chi(X) \quad . \tag{2.2}$$

This can be seen in several ways [22], for instance by using the relation between the Witten index and the index of the de Rham complex of X. The partition function $Z(S_X)$ can also be evaluated explicitly to give a path integral proof of the Gauss-Bonnet theorem which expresses $\chi(X)$ as an integral over X of the Pfaffian of the curvature form $\mathcal{R}_X$ of X,

$$Z(S_X) = \chi(X) = \int_X \mathrm{Pf}(\mathcal{R}_X) \ . \tag{2.3}$$

The crucial fact responsible for the reduction of the integral over the loop space LX of X (the path integral) to an integral over X (the Gauss-Bonnet integral) is the (topological) supersymmetry which ensures that only the Fourier zero modes (e.g. $\dot{x} = 0$) of the fields are relevant, the contributions from the other modes cancelling identically between the bosonic and fermionic fields.

It is the analogue in infinite dimensions of this observation that permits one to construct topological gauge theories in d (instead of $(d+1)$) dimensions from supersymmetric quantum mechanics on the space $\mathcal{A}^d/\mathcal{G}^d$ of gauge orbits of connections in d dimensions.

As they stand, the partition function and the right hand side of (2.3) do not make sense for infinite dimensional target spaces. There are, however, two refinements of the action (2.1) which turn out to have meaningful counterparts on $\mathcal{A}/\mathcal{G}$ [8]. The first of these involves a choice of potential $V(x)$ on X, the corresponding action being

$$S_{X,V} = \int dt\ g_{\mu\nu}(\dot{x}^\mu + g^{\mu\rho}\partial_\rho V(x))(\dot{x}^\nu + g^{\nu\sigma}\partial_\sigma V(x)) + \text{superpartners} \ . \tag{2.4}$$

The partition function $Z(S_{X,V})$ localizes to an integral over the set of critical points of V. There is also an intermediate localization [7, 11] which reproduces the finite-dimensional Mathai-Quillen formalism [9] (and hence that of [10, 1, 4] in the field theoretic setting).

In the case that the critical points of V are isolated and non-degenerate one arrives at the classical Poincaré-Hopf-Morse theorem

$$\chi(X) = \sum_{x_k : dV(x_k)=0} (\pm 1) \ . \tag{2.5}$$

If the critical points are not isolated then, by a combination of the arguments leading to (2.3) and (2.5), one finds

$$\chi(X) = \sum_{(k)} \chi(X_V^{(k)}) \ , \tag{2.6}$$

where the $X_V^{(k)}$ are the connected components of the critical point set of V and once again the sum is to be taken with signs depending on the relative orientations of the critical submanifolds induced by the gradient flow.

The relevance of this for our purposes is that the right hand side of (2.6) may be well defined, even if X is infinite dimensional, provided that X_V is finite dimensional. In that case $\chi(X_V)$ is well defined and can be regarded as a regularized Euler number of X (this is the point of view adopted in [10]). The advantage of the present construction is that it permits an *a priori* identification of this V-dependent regularized Euler number of X with the Euler number of X_V [7, 11].

Although this looks like a satisfactory state of affairs, we may not always be so fortunate to have a potential at our disposal whose critical points define precisely the (moduli) subspace $Y \subset X$ we are interested in. In fact, it follows from (2.6) that in finite dimensions $\chi(Y) = \chi(X)$ is a necessary condition for this to be possible.

To motivate the following construction, recall that the classical Gauss-Codazzi equations express the intrinsic curvature $\mathcal{R}_Y$ of Y (with the induced metric) in terms of $\mathcal{R}_X$ restricted to Y and a term quadratic in the extrinsic curvature K_Y of Y. The construction of an action $S_{Y\subset X}$ calculating $\chi(Y)$ will be modelled on this decomposition of $\mathcal{R}_Y$, i.e. it will consist of the action S_X (2.1) plus a supersymmetric Lagrange multiplier term enforcing the restriction to $Y \subset X$. Concretely, one introduces an $N_T\!=\!2$ coordinate superfield $X(t,\theta^m) = x(t) + \ldots$ (where $\theta^m = (\theta,\bar{\theta})$ are Grassmann odd scalars) and, assuming that Y is (locally) defined by $F^a(x) = 0$, $N_T = 2$ Lagrange multiplier fields $\Lambda_a(t,\theta^m) = \lambda_a(t) + \ldots$ and chooses the action to be

$$S_{Y\subset X} = S_X + \alpha \int dt \int d\theta\, d\bar{\theta} \Lambda_a(t,\theta,\bar{\theta}) F^a(X(t,\theta,\bar{\theta})) \ , \qquad (2.7)$$

so that the integration over the Λ_a imposes the superconstraints $F^a(X) = 0$. The argument leading to the elimination of the non-constant modes is not affected by the addition of this term and provided that there are no λ zero modes the evaluation of the partition function and the Gauss-Bonnet theorem applied to $\mathcal{R}_Y$ lead to

$$Z(S_{Y\subset X}) = \chi(Y) \ , \qquad (2.8)$$

now valid for arbitrary submanifolds $Y \subset X$ (not necessarily of the form X_V). As usual in topological field theories, the construction works equally well when the delta function constraint is replaced by a Gaussian, i.e. when

one adds a term $\sim \Lambda^2$ to the action. This is the generalization required to be able to apply supersymmetric quantum mechanics to spaces of connections in some sort of generality. In fact, as has been shown in [7, 8] and we will recall in the following, the extension to infinite dimensional target spaces like spaces of maps or metrics or connections is now fairly immediate.

2.2 Actions with Potentials and Action Potentials

Let us, for concreteness, consider the case of gauge theories where $X = \mathcal{A}$ or $X = \mathcal{A}/\mathcal{G}$, the space of connections on some manifold M. A metric on M induces a natural metric on $\mathcal{A}$ and then a metric $g_{\mathcal{A}/\mathcal{G}}$ on $\mathcal{A}/\mathcal{G}$ via the horizontal projectors $h_A = \mathrm{id} - d_A(d_A^* d_A)^{-1} d_A^*$. As a first step, in analogy with (2.1), one might like to consider the action

$$S_{\mathcal{A}/\mathcal{G}} = \int dt\; g_{\mathcal{A}/\mathcal{G}}(\dot{A}, \dot{A}) + \text{superpartners} \;. \tag{2.9}$$

However, this does not yet give a well-defined action. But let us now, in turn, consider the two refimements of the quantum mechanics action discussed above. In order to illustrate the field theoretic version of the action $S_{X,V}$ (2.4), we consider the space of gauge orbits $\mathcal{A}^3/\mathcal{G}^3$ in $d = 3$ dimensions. Then a natural candidate for V is the Chern-Simons functional $V_{CS}(A)$ (that this is not quite a well-defined function on $\mathcal{A}^3/\mathcal{G}^3$, because of large gauge transformations, is not a problem as only its derivative enters in the construction of the action which is a well-defined closed but not exact one-form on $\mathcal{A}^3/\mathcal{G}^3$). Then the counterpart of (2.4) is

$$S_{\mathcal{A}^3/\mathcal{G}^3, V_{CS}} = \int dt\; g_{\mathcal{A}^3/\mathcal{G}^3}(\dot{A} - *F_A, \dot{A} - *F_A) + \text{superpartners} \;. \tag{2.10}$$

A priori, this looks like a $d = (3+1)$ dimensional field theory, and indeed it can be shown [8] that this is precisely the action of Donaldson-Witten theory on a four-manifold of the form $M \times S^1$. However, we already know that all the time-dependent modes can be eliminated from this action, leaving us with a three-dimensional field theory of the form

$$S_{\mathcal{A}^3/\mathcal{G}^3, V_{CS}} \to \int_M F_A * F_A + \text{superpartners} \tag{2.11}$$

calculating the Euler number of the moduli space $\mathcal{M}^{(3)}$ of flat connections on M, the critical points of V_{CS}.

Of course, this procedure of constructing a field theory in d dimensions from another one in $d+1$ is somewhat indirect and the superfield formalism introduced in [7] is a way of bypassing the auxiliary $(d+1)$-dimensional field theory and constructing directly the resulting d-dimensional action. In the $N_T=2$ superfield language, the action then consists of a universal kinetic term for $N_T=2$ theories, corresponding to $S_{\mathcal{A}/\mathcal{G}}$ (in terms of the $N_T=2$ superfields to be discussed in section 3, this is simply the term $\int d\theta^1 d\theta^2 \mathcal{F}_1 * \mathcal{F}_2$, where $\mathcal{F}_m$ is given in (3.53)), and an $N_T=2$ Chern-Simons action, so that, in the parlance of [4], V_{CS} (or whatever potential one chooses) is the bosonic part of the action potential.

2.3 Supersymmetric Quantum Mechnics and the Cofield Construction

To illustrate the Gauss-Codazzi version (2.7) of supersymmetric quantum mechanics and its uses, let us try to construct a theory of flat connections in $d=2$. In this case, there is no potential constructed from the gauge fields alone whose critical points would make up the desired moduli space. However, we can extend the space of fields and consider 'potentials' of the form $\int BF_A$ where B is an adjoint scalar field. The $N_T=2$ supersymmetric extension of this is precisely the Lagrange multiplier construction of (2.7), and the resulting theory is described by the action

$$S_{\mathcal{M}\subset\mathcal{A}/\mathcal{G}} = S_{\mathcal{A}/\mathcal{G}} + \int dt\; BF_A + \text{superpartners} \tag{2.12}$$

This reduces firstly to the constant modes and thus to the superfield construction of [7] in which the first term of (2.12) is replaced by the universal kinetic term for $N_T=2$ theories and the time-dependence of the second term is dropped. It subsequently reduces to the desired moduli space $\mathcal{M}$ provided that there are no B moduli, i.e. provided that one has a 'vanishing theorem' [1]. In the present case, this vanishing theorem holds for irreducible conenctions as then the critical points of $\int BF_A$ are of the form $F_A = B = 0$. We see that this is precisely the 'cofield construction' employed in [1] and elaborated upon in [14, 4]. Once again, the superfield construction proposed in [7] provides a short-cut to constructing actions of this type.

In order to describe instanton moduli spaces in $d=4$, all that needs to be done is to replace the scalar B by a self-dual two-form B_+. In that case, there will be a vanishing theorem precisely when the instanton moduli

space is smooth. However, in order to describe theories which localize to the instanton moduli space under different circumstances, one can also consider deformations of the condition $F_+ = 0$ by the fields appearing in the $N_T = 2$ B_+-multiplet (which will then lead to $F_+ = 0$ under possibly different conditions). It turns out [1] that a certain topological twist of $N=4$ $d=4$ Yang-Mills theory (essentially constructed first by Yamron [16] and recalled in section 4.2 below) leads to a particular deformation of this kind and the corresponding vanishing theorems have been analyzed in detail in [1]. Taking this modification into account, it can be verified that the action resulting from (2.12) agrees term by term with the action constructed in [1].

For both types of theories, the extension to topological gravity or topological sigma models is immediate. This is also readily seen in terms of superfields for which the arguments in section 3 imply the equivalence with the constructions presented in [14, 4].

2.4 Counting with vs. Counting without Signs

We pause here to comment on one aspect of the above construction which we did not state as clearly as we should have in [7, 22] although it is particularly transparent from the supersymmetric quantum mechanics point of view. It concerns the question, which was one of the motivations for introducing the cofield construction in [1], 'when is one counting with and when is one counting without signs?'. This question arises when the moduli space $\mathcal{M}$ one is localizing onto, the counterpart of either $X_V \subset X$ or $Y \subset X$ in the quantum mechanics setting, is not connected.

In general, counting solutions of equations, e.g. zeros of vector fields or sections of other bundles, 'with relative signs' yields a topological invariant under suitable compactness conditions. The prime example of this is the Poincaré-Hopf theorem (see (2.5)) which expresses the Euler number of a manifold X as the signed sum of zeros of a vector field on X. In this case, relative signs will have to appear not because the total number of solutions is not a topological invariant (it may be), but simply for the equality with $\chi(X)$ to hold. We will constrast this below with the result (2.8) where $\chi(Y)$ is not equated to, say, $\chi(X)$, or some other topological invariant. In that case, relative signs need not (and will not) appear and under favourable conditions the path integral calculates the absolute number of solutions.

Likewise, in its generalization to non-isloated critical points (2.6), the Euler

numbers of the connected components of the critical point set X_V will enter with relative signs. These signs potentially appear because of the usual sign ambiguity of fermionic determinants (Pfaffians). While this leaves undetermined the overall sign, the relative signs, which depend on the relative orientations of the connected components $X_V^{(k)}$, can be determined by the spectral flow along trajectories of the (gradient) vector field connecting two connected components. Thus, $N_T = 2$ gauge theory actions based on $S_{X,V}$, in particular the three-dimensional theory of flat conenctions, count Euler numbers with relative signs.[3]

As discussed in detail in [1], there are cases where even counting solutions without signs may yield a topological invariant, The conditions for this to occur were phrased in [1] in terms of an extended set of fields and equations (the original fields supplemented by cofields, the equations for the cofields being - to first order in the cofields - the linearized original equations) and suitable vanishing theorems implying that for all solutions to the extended set of equations the cofields are zero. Because of the constraint on the equations for the cofields, the determinants fixing the signs of these solutions are all positive and one is thus counting solutions of the original equations without signs.

However, this is precisely the structure appearing in the Gauss-Codazzi (Lagrange multiplier) construction (2.7) where the Lagrange multipliers are the cofields and the Lagrangian is of the form $\lambda_a F^a + \ldots$ or $BF_A + \ldots$. Explicit evaluation of the determinants shows that all the fermionic determinants appear in pairs (because of the $N_T = 2$ structure) so that, for suitable vanishing theorems, one is counting Euler numbers without signs. This is particularly clear in the original example, action (2.7), where for a disconnected submanifold $Y \subset X$ one will simply obtain in (2.8) the sum of the Euler numbers $\chi(Y^{(k)})$ of its connected components, no relative signs possibly appearing as the partition function $Z(S_{Y\subset X})$ manifestly just reduces to a sum of the partition functions $Z(S_{Y^{(k)}})$ for actions of the original form (2.1).

Finally, let us point out that the above discussion shows that in addition to the $d = 3$ theory counting $\chi(\mathcal{M}^{(3)})$ with signs (the Casson invariant when the underlying three-manifold is a homology sphere) there is potentially another one, obtained on using Lagrange multipliers (or the cofield construction) to localize onto $\mathcal{M}^{(3)}$, which counts them without if appropriate deformations

[3]This played a crucial role in our suggestion [7] to regard $\chi(\mathcal{M}^{(3)})$ as a possible definition for a generalization of the Casson invariant away from homology spheres.

of the flatness condition and vanishing theorems could be found. This is just the dimensional reduction of the corresponding theory for instantons in $d = 4$ and its action would schematically be of the form $\int \mathcal{B} F_{\mathcal{A}}$ where $\mathcal{B}$ and $\mathcal{A}$ are $N_T = 2$ superfields of the type to be discussed in section 3.

3 $N_T\!=\!2$ Superfields and Equivariant Cohomology

The purpose of this section is to exhibit the relationship between the equivariant approach to $N_T = 2$ theories of Dijkgraaf and Moore [4], which includes the superfield version of Vafa and Witten [1], and what at first sight, appears to be be a completely different superfield approach employed in [7]. We will establish the relationship between the various formulations in the context of gauge theories. It should be apparent, however, that the equivalence is rather more general.

It is useful to summarise and compare the strategies that are involved. The approach of Dijkgraaf and Moore is to begin with the so called Weil model which requires the introduction of

$$\sum_{i=1}^{N_T} \left(\begin{array}{c} N_T \\ i \end{array} \right) = 2^{N_T} - 1 \tag{3.1}$$

'connections' and

$$(N_T - 1)2^{N_T} + 1 \tag{3.2}$$

independent 'curvatures'. The cohomology of interest is basic cohomology, that is one also demands that the representatives be gauge invariant (in an appropriate sense). Dijkgraaf and Moore then pass to the Cartan model where the connections are 'forgotten' and one deals solely with the curvatures. The price to be paid is that the cohomology is now 'equivariant', that is it closes only up to gauge transformations which are parameterised by the curvatures.

The strategy we employed was to consider superspaces of the form (x^μ, θ^m) where $m = 1, \ldots, N_T$, and the θ^m are Grassmann odd space time scalars and to consider a superfield $\mathcal{A}_m$ with the demand that it transforms as a connection. Given the connection one can form (linearly dependent) curvatures. The total number of independent fields is

$$N_T \times 2^{N_T} \tag{3.3}$$

which agrees with the total number of fields in the Weil model of [4]. The usual cohomology arises as supersymmetry transformations (shifts in the θ^m) and requiring 'super' gauge invariance. In order to pass to the Cartan model in this approach one sets $2^{N_T} - 1$ connections to zero, using up all of the 'super' gauge invariance that is available. The left-over terms in the superfield $\mathcal{A}_\theta$ are in fact curvatures. Supersymmetry transformations change the gauge so, in order to preserve the gauge, one needs to do a compensating super gauge transformation. The combined transformations are precicely the equivariant exterior derivatives of the Cartan model of Dijkgraaf and Moore.

As well as the new superfield $\mathcal{A}_m$, there will be the 'physical' or geometric superfield $\mathcal{A}_\mu$ whose zero'th order component is the gauge field A_μ. Depending on the theory at hand there may also be other multiplier superfields or topological matter superfields. However, for the general considerations of this section, we will not have to specify these in any detail.

The rest of this section is dedicated to filling in the details of these relationships for $N_T = 1$ and $N_T = 2$. We will allude to the general N_T theories but only briefly since the geometry they encode has still to be determined.

3.1 Superspace Formulation of $N_T = 1$ Gauge Theory

The superspace formulation that we will make use of in this section was introduced into topological field theory by Horne [23]. It was re-interpreted in [24] as a method for explaining the appearance of basic cohomology in topological field theories.

The superspace of interest here is (x^μ, θ), where θ is a Grassman-odd coordinate that is a space-time scalar. One expands fields in terms of θ, but since $\theta^2 = 0$, the expansion terminates at the second term. All tensor fields are now understood to have components also in the θ direction.

This means that supergauge fields are taken to have a Grassman component; i.e. a supergauge field is a pair $(\mathcal{A}, \mathcal{A}_\theta)$, where $\mathcal{A} = \mathcal{A}_\mu dx^\mu$, with the expansion[4]

$$\begin{aligned} \mathcal{A} &= A + \theta\psi \\ \mathcal{A}_\theta &= \xi + \theta\left(\phi - \frac{1}{2}[\xi,\xi]\right). \end{aligned} \tag{3.4}$$

[4]One can shift ϕ to $\phi' = \phi - \frac{1}{2}[\xi,\xi]$, but the form given in (3.4) is, as we shall see, canonical.

Supergauge transformations are

$$\begin{pmatrix} \mathcal{A} \\ \mathcal{A}_\theta \end{pmatrix} \to V^{-1} \begin{pmatrix} \mathcal{A} \\ \mathcal{A}_\theta \end{pmatrix} V + V^{-1} \begin{pmatrix} d \\ \partial_\theta \end{pmatrix} V \tag{3.5}$$

where

$$V = e^{\theta\lambda} g \tag{3.6}$$

and g is a conventional gauge transformation.

If we construct an action out of our superfields then supersymmetry is simply the statement that we can shift the grassman coordinate θ by ϵ without changing anything;

$$\int d\theta \mathcal{L}\left(\mathcal{A}(\theta+\epsilon)\right) = \int d\theta \mathcal{L}\left(\mathcal{A}(\theta)\right). \tag{3.7}$$

Furthermore, we consider an action which is invariant under the supergauge transformations defined above. In most instances this means that, up to the inclusion of auxiliary multiplier superfields, the action will be made out of the superfield curvatures (and covariant derivatives thereof) associated with the supergauge field. However, one can make use of a super-version of Chern-Simons theory in three dimensions as well in which case $\mathcal{A}$ itself will appear explicitly in the action.

The supersymmetry transformations then read

$$\begin{aligned} \delta\mathcal{A} &= \mathcal{A}(\theta+\epsilon) - \mathcal{A}(\theta) \\ \delta\mathcal{A}_\theta &= \mathcal{A}_\theta(\theta+\epsilon) - \mathcal{A}_\theta(\theta) \end{aligned} \tag{3.8}$$

or, in terms of components,

$$\begin{aligned} \delta A &= \psi \\ \delta\psi &= 0 \\ \delta\xi &= \phi - \frac{1}{2}[\xi,\xi] \\ \delta\phi &= [\phi,\xi]. \end{aligned} \tag{3.9}$$

This description of the theory is highly redundant. One can express the superfield $\mathcal{A}_\theta$ as

$$\mathcal{A}_\theta = U^{-1}\mathcal{A}_\theta^0 U + U^{-1}\partial_\theta U \tag{3.10}$$

where

$$\mathcal{A}_\theta^0 = \theta\phi, \tag{3.11}$$

$$U(\xi) = \exp(\theta\xi). \tag{3.12}$$

The virtue of writing things in this way is that we see immediately that *all* of the 'super' part of the supergauge transformations is taken up in the field ξ. This means that in a supergauge invariant action the field ξ will not appear since it is pure gauge. One sees this most clearly by considering the action of the super-part of a supergauge transformation. Under such a gauge transformation (i.e. $g = I$ in (3.6)) from (3.10) we see that

$$U(\xi) \to U(\xi)U(\lambda) = U(\xi + \lambda) \tag{3.13}$$

which corresponds to a shift of ξ, while ϕ is invariant

$$\begin{aligned} \xi &\to \xi + \lambda \\ \phi &\to \phi. \end{aligned} \tag{3.14}$$

The transformation (3.13) means that we can choose a gauge where there is no ξ component for $\mathcal{A}_\theta$. This gauge is achieved by taking

$$\lambda = -\xi \tag{3.15}$$

since in this case $U(\xi)U(-\xi) = 1$.

We should always be able to work with fields of the form (3.11) at the cost of only allowing conventional gauge transformations. This would be correct if we were not interested in supersymmetry as well. The problem is that under a supersymmetry transformation

$$\begin{aligned} \mathcal{A}_\theta(\theta) \to \quad & \mathcal{A}_\theta(\theta + \epsilon) \\ = \quad & \epsilon\phi + \theta\phi \end{aligned} \tag{3.16}$$

so that a ξ term $(= \epsilon\phi)$ reappears. However, we know that by a gauge transformation we can eliminate *any* ξ term. So the strategy is to follow the shift in θ by a gauge transformation that puts the superfield back into the form where there is no ξ component. This is easy to do and it is straightforward to determine the gauge transformation that is required (it is $\exp(-\theta\epsilon\phi)$) though in fact we will not need its explicit form in the following.

A consequence of this is that the action, with the reduced field content (that is with $\xi = 0$), will only be invariant under the combined shift and supergauge transformation (as well as conventional gauge transformations). We will deduce the appropriate symmetry by a slightly indirect method that,

nevertheless, will be useful when we come to comparing with the work of Dijkgraaf and Moore. We introduce the field strengths that are associated with the connections (3.4) (we have dropped the superscript 0)

$$\begin{aligned}
\mathcal{F}_A &= d\mathcal{A} + \mathcal{A}^2 \\
&= F_A + \theta d_A \psi \\
\mathcal{F}_\theta &= \partial_\theta \mathcal{A} - d\mathcal{A}_\theta - [\mathcal{A}, \mathcal{A}_\theta] \\
&= \partial_\theta \mathcal{A} - d_{\mathcal{A}} \mathcal{A}_\theta \\
&= \psi - \theta d_A \phi \\
\mathcal{F}_{\theta\theta} &= 2\partial_\theta \mathcal{A}_\theta + \mathcal{A}_\theta^2 \\
&= 2\phi.
\end{aligned} \tag{3.17}$$

Notice that ϕ from (3.17), though it appears in $\mathcal{A}_\theta$, is indeed a curvature[5].

To deduce the transformations, we note that for all the field strengths, under the combined shift and gauge transformation we have

$$\mathcal{F}(\theta) \to U^{-1}(\theta, \epsilon)\mathcal{F}(\theta + \epsilon)U(\theta, \epsilon), \tag{3.18}$$

where $U(\theta, \epsilon)$ is the gauge transformation which puts the superfield back into the form (3.11) and necessarily $U(0, \epsilon) = I = U(\theta, 0)$. Now we will argue that one can safely ignore $U(\theta, \epsilon)$ in determining the transformation properties of the fields. The argument is as follows: we wish to determine the transformation of the fields that appear at $O(1)$ in the field strengths, that is ψ and ϕ but, since $U = I + O(\theta)$, U can only contribute to terms of order θ so we may safely for present purposes set it to unity. We therefore, when dealing with the curvatures, only need to consider shifts in θ. From (3.17) we easily deduce that the topological transformation is

$$\begin{aligned}
QA &= \psi \\
Q\psi &= -d_A \phi \\
Q\phi &= 0
\end{aligned} \tag{3.19}$$

(we obtained the A transformation using the same logic but applied to $\mathcal{A}$). Let us repeat that we have derived these transformations without having to know the explicit form of $U(\theta, \epsilon)$.

[5] We have been continually referring to ξ as a connection in the superfield approach. The reason for this nomenclature is that if one calculates curvatures prior to gauge fixing then ξ, like A, never appears as the lowest component of a curvature superfield. For example $\mathcal{F}_\theta = \psi - d_A \xi + O(\theta)$.

3.2 Weil and Cartan Models

We will now relate the above construction to something that appears in the mathematics literature. For G a Lie group and $\mathfrak{g}$ its algebra, we define the Weil algebra $\mathcal{W}(\mathfrak{g})$ by

$$\mathcal{W}(\mathfrak{g}) = \Lambda(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*) \tag{3.20}$$

which is the tensor product of the exterior algebra and the symmetric algebra of the dual $\mathfrak{g}^*$. Denote the generator of $\Lambda(\mathfrak{g}^*)$ by ξ with degree 1 (it is a connection) and the generator of $S(\mathfrak{g}^*)$ of degree 2 (a field strength) by ϕ. On the Weil algebra one can define a covariant derivative $d_W + \xi$ where d_W is a derivation on $\mathcal{W}$. The relationship between the various objects is

$$\phi = \frac{1}{2}[d_W + \xi, d_W + \xi], \quad d_W\phi + [\xi, \phi] = 0 \tag{3.21}$$

the second equation being the Bianchi identity. These are rewritten as

$$\begin{aligned} d_W\xi &= \phi - \frac{1}{2}[\xi, \xi] \\ d_W\phi &= [\phi, \xi]. \end{aligned} \tag{3.22}$$

The cohomology of the Weil complex is trivial since, by shifting $\phi' = \phi - [\xi, \xi]/2$, one finds

$$d_W\xi = \phi', \quad d_W\phi' = 0. \tag{3.23}$$

The cohomology of interest is actually the so called 'basic' cohomology. The interior derivative i_a is defined by

$$i_a\xi^b = \delta^b_a, \quad i_a\phi^b = 0, \tag{3.24}$$

A form is called basic if it is horizontal and gauge invariant, i.e. it satisfies

$$i_a\omega = 0, \quad \mathcal{L}_a\omega = 0 \tag{3.25}$$

where

$$\mathcal{L}_a = [d_W, i_a]. \tag{3.26}$$

In this context it is useful to pass to the Cartan model. Formally, this amounts to 'forgetting' ξ. This means that one sends $\xi \mapsto 0$ and considers this as a map of $\mathcal{W}(\mathfrak{g}^*) \to S(\mathfrak{g}^*)$ which induces an algebra isomorphism

$$\mathcal{W}(\mathfrak{g}^*)_{\text{basic}} \equiv S(\mathfrak{g}^*)^G \tag{3.27}$$

where the superscript G means the G-invariant subalgebra. This is an isomorphism of differential (graded) algebras when one defines the derivation

$$d_C \phi = 0 \tag{3.28}$$

on $S(\mathfrak{g}^*)^G$.

Let us generalise a little. Consider the algebra $\mathcal{W}(\mathfrak{g}^*) \otimes \Omega^*(M)$, where M is some manifold on which G acts. Define the derivation on this algebra by

$$d^W = d_W \otimes 1 + 1 \otimes d. \tag{3.29}$$

The basic forms in this case are defined by, for $\omega \in \mathcal{W}(\mathfrak{g}^*) \otimes \Omega^*(M)$,

$$i_a \omega = \mathcal{L}_a \omega = 0. \tag{3.30}$$

To pass to the Cartan model we have to ask what the effect of the 'forgetting ξ' is on the extended space. The result is that one defines the Cartan derivation in this case as

$$d^C = 1 \otimes d - \phi^a \otimes i_a. \tag{3.31}$$

3.3 Equivalence of the Weil, Cartan and Superfield Models

We begin with the trivial observation that the field content, (ξ, ϕ), of the Weil and the superfield models are the same. Indeed the fields have the same interpretation, ξ is a connection while ϕ is a curvature. Secondly, the action of the exterior derivative d_W in the Weil model (3.22) agrees with the supersymmetric shift operation (3.9), so we conclude that

$$\delta = d_W \tag{3.32}$$

In order to pass to the basic cohomology in the Weil model one needs an inner derivation. By comparing (3.24) and (3.14) we see that what are called super gauge transformations $\exp(\theta\lambda)$ correspond precicely to the derivations $\lambda^a i_a$. In other words basic forms are, in the superfield language, gauge invariant forms.

The isomorphism between the Weil and the Cartan models, as graded differential algebras, only works when one restricts ones attention to basic forms in the Weil model since on setting $\xi = 0$ we would deduce from (3.21) that $\phi = 0$. In the time honoured tradition of gauge theory we either work equivariantly or we gauge fix. Working equivariantly in the Weil model means

working with basic forms while gauge fixing means working with the Cartan model. The gauge we would like to choose is $\xi = 0$. We saw above that if we do this then we would be led to

$$\delta 0 = d_W 0 = \phi \tag{3.33}$$

and that a way out is to supplement the derivation $\delta = \partial/\partial\theta$ with a gauge transformation, and we called the new derivation Q, so that

$$Q\mathcal{A}_\theta = 0 \tag{3.34}$$

and we identify, on $S(\mathfrak{g}^*)^G$,

$$d_C = Q, \tag{3.35}$$

(and setting ξ to zero is now fine since the derivation equation is $d_C\xi = 0$). Somewhat more generally the action of the group G is understood to be simultaneously an action on both factors of $\mathcal{W}(\mathfrak{g}^*)\otimes\Omega^*(M)$. d_C was obtained by supplementing d_W with a gauge transformation, i.e. $d_C = d_W - \phi^a i_a$. A natural derivation is that obtained on gauge transforming both differentials. So we are led to

$$d^C = d_C \otimes 1 + 1 \otimes d - \phi^a \otimes i_a \tag{3.36}$$

which is the standard differential in the Cartan model (the first term gives zero).

Now, what this translates to for gauge theories, in the 'geometric' sector is, since the group that acts is the group of gauge transformations,

$$d^C A = 0, \quad d^C\psi = d_A\phi, \quad d^C\phi = 0. \tag{3.37}$$

These agree with the action of Q, but this comes as no surprise since we have seen in the previous section that this is *exactly* how the BRST operator arises in the $N_T = 1$ gauge theory. In fact the relationship between Q and d^C is rather transparent when one looks at the curvature $\mathcal{F}_\theta$ (3.17) which we rewrite as

$$\mathcal{F}_\theta = \left(\frac{\partial}{\partial\theta} - i_{\mathcal{A}_\theta}\right)\mathcal{A}. \tag{3.38}$$

It is also worthwhile exhibiting the relationship by determining U, since this approach will facilitate the comparison for other values of N_T. We have commented before that

$$U = e^{-\theta\epsilon\phi} = e^{-\theta\mathcal{A}_\theta(\epsilon)} \tag{3.39}$$

so that to first order in ϵ the definition of Q is: perform a translation, i.e. act by $1 \otimes d$, and follow that by a gauge transformation $U = 1 - \theta\epsilon\phi$, i.e. act by $-\phi^a \otimes i_a$.

Remarks:

1. The relationship with the paper of Dijkgraaf and Moore is clear. Their connection ω is what we have called ξ and we agree on the symbol for the curvature ϕ.

2. In the superfield approach both the connection ξ and the curvature ϕ appear in the same superfield. This happens because, from this point of view, curvatures are not independent of connections.

3. One does not even need to know the transformations (3.19) to construct an invariant action. On using the curvature superfields we are assured that we will have an action which is Q invariant and Q exact. Furthermore, without knowing what Q is we know that $Q^2 = \mathcal{L}_\lambda$, for some λ, by construction. It is this level of generality which makes the superfield approach easy to use.

4. There is also a well known procedure for arriving at topological observables. One constructs characteristic classes of the super curvatures.

5. In order to determine the transformations of multiplier superfields, once ξ has been set to zero, one uses the same logic that we employed on the curvatures, but this time on covariant derivatives of the multiplier fields.

3.4 Superfield Formulation of $N_T = 2$ Gauge Theory

The superspace in question is that spanned by two Grassmann coordinates which we denote by θ^m, $m = 1, 2$. Some notation is required. Let

$$\theta^2 = \frac{1}{2}\epsilon_{mn}\theta^m\theta^n, \quad \epsilon_{12} = 1, \tag{3.40}$$

then we find that

$$\theta^m\theta^n = \epsilon^{mn}\theta^2, \quad \epsilon^{12} = 1. \tag{3.41}$$

There are two Grassmann components of the gauge field which are denoted by $\mathcal{A}_m$. As for the $N_T = 1$ theory a meaningful way to represent the super

gauge field is as follows

$$\mathcal{A}_m = U^{-1}\mathcal{A}^0_m U + U^{-1}\partial_m U \tag{3.42}$$

where

$$\mathcal{A}^0_m = \theta^n \phi_{nm} + \theta^2 \eta_m, \tag{3.43}$$

$$U = \exp\Big(\theta^m \xi_m + \theta^2 \rho\Big), \tag{3.44}$$

and $\phi_{mn} = \phi_{nm}$. The ξ_n and ρ are connections and the ϕ_{nm} and η_m are (generalised) curvatures. We note that the connections match with the fields $\omega_\pm$ and Ω of Dijkgraaf and Moore, as do the curvatures which bear the same name (cf. [4, eq (3.16)]).

The Grassmann part of the gauge group has three independent variables

$$U = \exp\Big(\theta^m \lambda_m + \theta^2 \sigma\Big) = 1 + \theta^m \lambda_m + \theta^2(\sigma - \lambda^2), \tag{3.45}$$

with $\lambda^2 = \frac{1}{2}\epsilon^{mn}\lambda_m\lambda_n$. Under such a super-gauge transformation we have

$$\mathcal{A}_m \to U^{-1}\mathcal{A}_m U + U^{-1}\partial_m U, \tag{3.46}$$

or

$$\exp\Big(\theta^m \xi_m + \theta^2 \rho\Big) \to \exp\Big(\theta^m \xi_m + \theta^2 \rho\Big)\exp\Big(\theta^m \lambda_m + \theta^2 \sigma\Big) \tag{3.47}$$

from which we deduce that

$$\begin{aligned} \delta\xi_m &= \lambda_m \\ \delta\phi_{nm} &= 0 \\ \delta\rho &= \sigma + \frac{1}{2}\epsilon^{nm}[\lambda_n, \xi_m] \\ \delta\eta_m &= 0. \end{aligned} \tag{3.48}$$

Since the gauge field $\mathcal{A}_m$ is the gauge transform of $\mathcal{A}^0_m$ we need only calculate the curvature tensors for $\mathcal{A}^0_m$. We will drop the superscript. The usual gauge field A is the first component of a superfield

$$\mathcal{A} = A + \theta^m \psi_m + \theta^2 B. \tag{3.49}$$

The general definition of the curvature tensor for a superspace with N_T scalar grassmann coordinates in d dimensions is

$$\mathcal{F}_{MN} = \partial_M \mathcal{A}_N - [MN]\partial_N \mathcal{A}_M + [\mathcal{A}_M, \mathcal{A}_N] \tag{3.50}$$

with $M,N = (\mu,m),(\nu,n)$ where $\mu,\nu = 1,\dots,d$ and $m,n = 1,\dots,N_T$. The symbol $[MN] = \pm 1$, is -1 only if $M = m$ and $N = n$ simultaneously.

One finds that, with $N_T = 2$,

$$\mathcal{F}_{mn} = 2\phi_{mn} + \theta^p(\epsilon_{mp}\eta_n + \epsilon_{np}\eta_m) + O(\theta^2) \tag{3.51}$$

and

$$\epsilon^{mp}[D_p, \mathcal{F}_{mn}] = 3\eta_n - 3\theta^m\epsilon^{rs}[\phi_{rm}, \phi_{sn}] + O(\theta^2) \tag{3.52}$$

$$\mathcal{F}_m = \psi_m + \epsilon_{mn}\theta^n B - \theta^n d_A\phi_{nm} + O(\theta^2) \tag{3.53}$$

$$\epsilon^{mn}D_n\mathcal{F}_m = 2B + \theta^m d_A\eta_m + \epsilon^{mn}\theta^p[\psi_n, \phi_{pn}] + O(\theta^2). \tag{3.54}$$

Using the fact that curvatures transform homogeneously and that U is necessarily of order θ we deduce from the curvatures above that

$$\begin{aligned}
Q_nA &= \psi_n \\
Q_n\psi_m &= \epsilon_{mn}B - d_A\phi_{mn} \\
Q_nB &= \frac{1}{2}d_A\eta_n + \frac{1}{2}\epsilon^{mp}[\psi_p, \phi_{mn}] \\
Q_n\phi_{mp} &= \frac{1}{2}\epsilon_{mn}\eta_p + \frac{1}{2}\epsilon_{pn}\eta_m \\
Q_n\eta_m &= \epsilon^{pq}[\phi_{nq}, \phi_{mp}].
\end{aligned} \tag{3.55}$$

These transformations agree with those of [1] and [4] (taking into account that their η_m and ours differ by a factor of two). In [7] we gave a different method for determining these transformations which amounts essentially to fixing U directly. We will follow a similar course in the next section when we come to comparing with [4].

3.5 Equivalence of the Weil, Cartan and Superfield Models for $N_T = 2$

Rather than reviewing each model seperately we can proceed directly to a comparison. The Weil model à la Dijkgraaf and Moore has three connections (ξ_m, ρ) and field strengths (ϕ_{nm}, η_m). These fields are the components of $\mathcal{A}_m$ in the superfield language. As for the $N_T = 1$ theories we would like to pass from the Weil model to the Cartan, and as before we do this by gauge fixing the 'connections' (ξ_m, ρ) to zero (as we have done for the superfields). Next

we identify the derivations. In the equivariant cohomology of Dijkgraaf and Moore there are derivations i^m and I which satisfy the algebra

$$[i^m(V), i^n(W)] = \epsilon^{pm} I\left([W, V]\right) \tag{3.56}$$

with other (anti-) commutators being zero. They act on the connections by

$$i^m_a \xi^b_n = \delta^m_n \delta^b_a, \quad i^m_a \rho^b = \frac{1}{2}\epsilon^{mn}[T_a, \xi_n]^b \tag{3.57}$$

and

$$I_a \xi^b_m = 0, \quad I_a \rho^b = \delta^b_a. \tag{3.58}$$

In our model these correspond to the possible 'super' gauge transformations. By inspecting the supersymmetry transformations (3.14) we see that the λ_m part of the gauge transformation corresponds to i^m while in the σ part one recognises the action of I. The algebra that they satisfy is easily deduced by composing two superfield gauge transformations whose exponents depend only θ^m and writing the product as a single superfield gauge transformation which has in the exponent a term proportional to θ^2. The algebra of course agrees with (3.56).

Now the definition of the Cartan derivations on the physical fields, following [4], is

$$d^C_m = d^W_m + \phi^a_{mn} i^n(V_a) + \eta^a_m I(V_a) \tag{3.59}$$

which agrees with the action of Q_m. However, it is again enlightning to see this in terms of U. To determine U, we note that what we would like is the solution to the equations

$$\begin{aligned} \mathcal{A}'_m|_{\theta=0} &= \mathcal{A}_m(\epsilon) + \partial_m U(\theta,\epsilon)|_{\theta=0} = 0 \\ \epsilon^{mn}\partial_n \mathcal{A}'_m|_{\theta=0} &= \epsilon^{mn}\left(\partial_m U(\theta,\epsilon)|_{\theta=0}\partial_n U(\theta,\epsilon)|_{\theta=0}\right. \\ &\quad \left. + \partial_n \mathcal{A}_m(\epsilon) + \partial_n\partial_m U(\theta,\epsilon)|_{\theta=0}\right) = 0 \end{aligned} \tag{3.60}$$

The solution to these equations is

$$U = \exp\left(-\theta^m \mathcal{A}_m(\epsilon) - \frac{\theta^2}{2}\epsilon^{mn}\partial_n \mathcal{A}_m(\epsilon)\right). \tag{3.61}$$

We have already identified the θ term of a gauge transformation as the coefficient of a derivation i^m and the θ^2 term as the coefficient of the derivation

I so that by inspection of (3.61) and to first order in ϵ we have that U acts by

$$-\epsilon^m \left[\phi^a_{mn} i^n_a - \frac{1}{2}\eta^a_m I_a\right] \tag{3.62}$$

once more in complete agreement (up to the ubiquitous factor of 2 in front of η) with [4].

Most of the remarks that we made for the $N_T = 1$ models hold here as well with slight modification. In particular, any action made out of the superfield curvatures will be $\epsilon^{mn} Q_m Q_n$ exact.

One peculiarity of three dimensions is that there the 'smallest' topological gauge theory appears to have $N_T = 2$. One could construct, formally, an $N_T = 1$ super-BF theory, but it is easy to see that this action agrees with that which one obtains from the $N_T = 2$ Chern-Simons action. This also explains why the partition function comes in with signs. We will establish a more general result (for arbitrary N_T) in the next section.

In summary, we have seen that the algebraic constructions of Vafa and Witten [1] and of Dijkgraaf and Moore [4] are indeed identical to the geometric superfield approach that we introduced in [7], the two points of view being complementary. A variant on this theme preserving the underlying $sp(2,\mathbb{R})$ symmetry and which does not set the connections to zero (corresponding to an $N_T = 2$ version of the so-called BRST model [25] of equivariant cohomology) can be found in [26].

3.6 Arbitrary N_T

As Dijkgraaf and Moore briefly consider the extension of their formalism to arbitrary N_T, let us indicate how this can be done in terms of superfields.

An arbitrary N_T gauge field has as an expansion

$$\mathcal{A}_m = \sum_{i=0}^{N_T} \theta^{j_1} \dots \theta^{j_i} \Phi_{j_1 \dots j_i, m} \tag{3.63}$$

the components of which are products of a totally antisymmetric tensor in i labels and the fundamental vector N_T. Such a product can always be written as the direct sum of a $i+1$ totally antisymmetric tensor and a tensor of mixed symmetry (with Young tableau consisting of two columns, the first has i rows the second just one row).

Gauge parameters have an expansion in terms of forms (i.e. totally antisymmetric tensors), so that

$$U = e^{\Lambda} \tag{3.64}$$

with

$$\Lambda = \sum_{i=1}^{N_T} \theta^{j_1} \dots \theta^{j_i} \rho_{j_1 \dots j_i} \tag{3.65}$$

We conclude then that any $\mathcal{A}_m$ can be written as the gauge transform of

$$\mathcal{A}^0_m = \sum_{i=0}^{N_T} \theta^{j_1} \dots \theta^{j_i} \Phi_{j_1 \dots j_i m}, \tag{3.66}$$

where

$$\Phi_{j_1 \dots j_i m} \tag{3.67}$$

is the tensor of mixed symmetry.

How many derivations do we have in this model? The answer is that the number of derivations matches the number of 'super' gauge parameters, so we have the derivations

$$i_m,\ i_{mn},\ \dots, \epsilon_{m_1, m_2, \dots m_{N_T}} I. \tag{3.68}$$

One can, once more, determine the algebra of these by expanding the superfields.

If one wishes to make contact with the work of [4] then one can straightforwardly determine the appropriate U, bearing in mind that one requires it only to first order in ϵ (higher orders tell one about successive BRST transformations). One now wants to solve

$$\mathcal{A}'_m|_{\text{tas}} = \Big(U^{-1} \mathcal{A}_m(\theta + \epsilon) U + U^{-1} \partial_m U \Big) |_{\text{tas}} = 0 \tag{3.69}$$

where tas stands for the totally antisymmetric tensor part. Writing U to first order in ϵ, as

$$U = 1 + \epsilon^m \Lambda_m(\theta) + \dots \tag{3.70}$$

(3.69) becomes

$$[\mathcal{A}_m(\theta), \epsilon^n \Lambda_n]|_{\text{tas}} + \epsilon^n \partial_n \mathcal{A}_m|_{\text{tas}} - \epsilon^n \partial_m \Lambda_n|_{\text{tas}} = 0, \tag{3.71}$$

which can be solved order by order in θ.

While to date there has been no application of these types of topological field theories with $N_T \geq 3$ and their meaning remains unclear, in three dimensions a possible action for $N_T = 2n$ is the super Chern-Simons functional. This does not work for $N_T = 2n+1$ since the top component of $\mathcal{A}$ is Grassmann odd in that case.

We will now show that $\mathcal{BF_A}$ theory in 3 dimensions with $N_T = 2n-1$ is the same as the superfield Chern-Simons action for $N_T = 2n$, thus eliminating the possibility of odd numbers of supersymmetries in this dimension (unless, of course, one breaks the symmetry down by hand). Let $N_T = 2n-1$ and let the Grassmann coordinates be θ^m with $m = 1, \ldots, N_T$. For the action to make sense the lowest component of $\mathcal{B}$ must be Grassmann odd. Introduce a new Grassmann variable $\theta = \theta^{N_T+1}$ and let

$$\mathcal{A}(\theta^m, \theta) = \mathcal{A}(\theta^m) + \theta \mathcal{B}(\theta^m) \tag{3.72}$$

This is the most general expansion of a gauge superfield in θ, and so this is the most general expression of a gauge superfield in a superspace with $2n$ Grassmann coordinates. The Chern-Simons action is,

$$\prod_{i=1}^{N_T+1} \int d\theta^i S_{CS}\left(\mathcal{A}(\theta^m, \theta)\right) = \prod_{i=1}^{N_T} \int d\theta^i \mathcal{BF_A}. \tag{3.73}$$

Therefore there is an increase in the amount of supersymmetry in the theory.

The even N_T super-BF theories in 3 dimensions, on the other hand, can also be realised as super Chern-Simons theories with (a priori) the same N_T, but with the gauge group being IG, the tangent bundle group of G. The most well known example of this is the case $N_T = 0$ where the fields A and B of the G-BF theory can be combined into a connection $A + B$ for IG.

4 Other $N_T\!=\!2$ Topological Gauge Theories

In this section, we will briefly describe two other topological gauge theories with an extended $N_T\!=\!2$ topological superysmmetry which do not fall into the pattern of $N_T = 2$ theories described above. It is well known that extended supersymmetries can arise when $N_T = 1$ theories are formulated on manifolds with reduced holonomy groups (e.g. Donaldson-Witten theory on Kähler manifolds). The extended supersymmetries we will encounter below bear some resemblance to this, resulting however, roughly speaking,

from a complexification of the space of fields and not from, say, a Kähler stucture on the space-time manifold.

As a preparation, in section 4.1 we recall the structure of super-BF theories in $d=3,4$ [17, 6] and construct the quantum action using a minimal field content not nearly filling out the Batalin-Vilkovisky triangles by exploiting the topological supersymmetry present in these models.

In section 4.2, we show that the $N_T=2$ theory constructed by Marcus [13], the B-twist of $N=4$ $d=4$ Yang-Mills theory, can be regarded as a deformation of the four-dimensional $N_T=1$ super-BF theory [17, 6], but we refrain from a detailed discussion of that model as it is not clear that the study of moduli spaces of flat connections in $d=4$ is particularly meaningful.

In section 4.3 we describe a novel twist of $N=4$ $d=3$ Yang-Mills theory. This twist uses the internal $SU(2)$ arising in the reduction of $N=1$ $d=6$ Yang-Mills theory to $d=3$ and leads to some unusual features which we briefly describe.

4.1 Simplified Quantization of Super-BF Theories

Super-BF theories are cohomological gauge theories of flat connections. They can be defined in any dimension d, and the action typically takes the form $\int BF_A$ plus superpartners plus gauge fixing terms, where F_A is the curvature of the connection A, the mutiplier field B is a $(d-2)$-form in the adjoint of the gauge group and a trace is understood. The $N_T=1$ superpartners are $QA=\psi$, $Q\chi=B$, which will always be supplemented by an anti-ghost multiplier pair $(\bar{\phi},\eta)$ for ψ with $Q\bar{\phi}=\eta$, $Q\eta=[\bar{\phi},\phi]$ so that the 'classical' action can be taken to be

$$\begin{aligned} S_c^{(d)} &= Q\int \chi F_A + \bar{\phi} d_A * \psi \\ &= \int BF_A + (-1)^d \chi d_A\psi + \eta d_A * \psi + \bar{\phi}[\psi, *\psi] - \bar{\phi} d_A * d_A\phi \end{aligned} \tag{4.1}$$

As reviewed in detail in [6], the universal terms arising from the variation of the second term encapsulate concisely the geometry of the universal bundle over the space of gauge orbits - restricted to the moduli space of interst by the terms arising from the variation of the first term.

This action has the tower $\delta B = d_A B^{(1)}$, $\delta B^{(1)} = d_A B^{(2)}$, $\ldots$, of on-shell reducible symmetries and its quantization was analysed in detail in [17, 6]

using the Batalin-Vilkovisy procedure. In the simplest non-trivial case $d=3$ this leads to the following field content and action:

In addition to the universal geometric sector $(A, \psi, \phi, \bar{\phi}, \eta)$ and the multiplier pair (χ, B) one has scalar ghost-antighost-multiplier triplets $(\sigma, \bar{\sigma}, \pi)$ and $(\Sigma, \bar{\Sigma}, \Pi)$ for χ and B respectively. The off-shell equivariantly nilpotent Q-transformations are

$$\begin{aligned} QA &= \psi & Q\psi &= d_A\phi \\ Q\phi &= 0 & & \\ Q\bar{\phi} &= \eta & Q\eta &= [\bar{\phi}, \phi] \\ Q\chi &= B + d_A\sigma & QB &= d_A\Sigma + [\chi, \phi] + [\psi, \sigma] \\ Q\sigma &= \Sigma & Q\Sigma &= [\sigma, \phi] \\ Q\bar{\sigma} &= \pi & Q\pi &= [\bar{\sigma}, \phi] \\ Q\bar{\Sigma} &= \Pi & Q\Pi &= [\bar{\Sigma}, \phi] \end{aligned} \tag{4.2}$$

and the complete quantum action can be chosen to be

$$S_q^{(3)} = Q \int \chi F_A + \bar{\phi} d_A * \psi + \bar{\sigma} d_A * \chi + \bar{\Sigma} d_A * B \quad . \tag{4.3}$$

However, due to the Q-pairing there is clearly some redundancy in this description as e.g. the variation of the third term all by itself already provides a gauge fixing for χ and B. Moreover, by a field redefinition of B the ghosts σ and Σ can be eliminated from the picture.

This reasoning suggests an alternative procedure, bypassing the BV algorithm. Instead of two ghost-triplets one introduces just one antighost-multiplier pair $(\bar{\sigma}, \pi)$ which we will now call $(u, \bar{\eta})$ and postulates the equivariantly nilpotent algebra

$$\begin{aligned} QA &= \psi & Q\psi &= d_A\phi \\ Q\phi &= 0 & & \\ Q\bar{\phi} &= \eta & Q\eta &= [\bar{\phi}, \phi] \\ Q\chi &= B & QB &= [\chi, \phi] \\ Qu &= \bar{\eta} & Q\bar{\eta} &= [u, \phi] \end{aligned} \tag{4.4}$$

The corresponding action for this reduced field content,

$$S_q^{(3),red} = Q \int \chi F_A + \bar{\phi} d_A * \psi + u d_A * \chi \quad , \tag{4.5}$$

can readily be seen to be equivalent to (4.3) by integrating out the superfluous fields.[6]

Note that the field content suggests a discrete symmetry exchanging η and $\bar{\eta}$ and χ and ψ, so that there should actually be a (presently hidden) $N_T = 2$ symmetry in this theory. This is indeed the case. In fact, this theory is precisely the $N_T = 2$ theory (2.11) calculating the Euler characteristic of the moduli space of flat connections which we have discussed in detail in [7]. The reduced field content is that one obtains after eliminating the super-gauge transformation. In order to obtain a manifestly $N_T = 2$ symmetric action, one needs to add to (4.5) a term of the form $Q \int \chi * B$, thus regularizing the harmonic modes of B, and then integrate out B.

As we will recall below, this is also one of the actions one can obtain by twisting the $N = 4$ $d = 3$ Yang-Mills theory. From this point of view, the $N_T = 2$ symmetry is manifest as the fermions $(\eta, \bar{\eta})$ and (χ, ψ) then appear as doublets of the $SU(2)$ ghost-number symmetry of the theory. As $SU(2)$ is simple, this ghost number is not anomalous and the partition function is non-zero.

The four-dimensional super-BF theory has been discussed at length in [17, 6]. Its quantization is non-trivial as the χ-symmetry is now on-shell reducible, leading to the ghost-for-ghost phenomenon and cubic ghost interaction terms. Following the BV algorithm, one obtains an impressive field content consisting of a ghost and ghost for ghost for both χ and B and three antighost-multiplier pairs each, in addition to the fields $(A, \psi, \phi, \bar{\phi}, \eta)$. The corresponding quantum action, with an off-shell equivariantly nilpotent Q-symmetry, is spelled out in [6, eq. (5.133)].

In this case, the procedure outlined above for arriving at a reduced field content leads to a significant simplification. The transformations on the fields $(A, \psi, \phi, \bar{\phi}, \eta, \chi, B)$ remain as in (4.4). As χ and B are now two-forms, instead of the pair of scalars $(u, \bar{\eta})$ we introduce a pair of one-forms $(V, \bar{\psi})$ as well as the ghost-for-ghosts $(\bar{\eta}, u)$ (to account for the symmetries of (V, ψ)) with

$$\begin{aligned} QV &= \bar{\psi} \qquad & Q\bar{\psi} &= [V, \phi] \\ Q\bar{\eta} &= u \qquad & Qu &= [\bar{\eta}, \phi] \ . \end{aligned} \tag{4.6}$$

[6]To be precise, this holds modulo harmonic modes of the ghost-fields which have to be dealt with in some manner in the original action anyway.

The quantum action is now taken to be

$$S_q^{(4),red} = Q \int \chi F_A - \bar{\phi} d_A * \psi + V d_A * \chi - \bar{\eta} d_A * V \tag{4.7}$$

and clearly all the local symmetries of the action apart from ordinary gauge invariance have been gauge-fixed. The appearance of both ψ and $\bar{\psi}$ once again suggests the presence of a hidden $N_T = 2$ symmetry, although of a different kind this time as both ψ and $\bar{\psi}$ have ghost-number one. It is the aim of the next section to expose this symmetry and to show that the action can be deformed to that of the B-twist [13] of $N=4$ $d=4$ Yang-Mills theory.

4.2 Super-BF and the B-Twist of $N=4$ $d=4$ Yang-Mills Theory

In [5] Witten pointed out that the topological Yang-Mills theory he constructed (now known as Donaldson-Witten theory) could be regarded as a twisted version of $N=2$ $d=4$ Yang-Mills theory. The 'twist' involves replacing one factor of the Lorentz group $SU(2)_L \times SU(2)_R$ by its diagonal product with the internal R-symmetry group $SU(2)_I$. Thus, the fermions, which originally transform as

$$\text{Fermions:} \quad (2,1,2) \oplus (1,2,2) \tag{4.8}$$

under $SU(2)_L \times SU(2)_R \times SU(2)_I$, in the twisted theory theory transform as, say,

$$\text{Fermions} \rightarrow (2,2) \oplus (1,1) \oplus (1,3) \quad , \tag{4.9}$$

leading to a Grasssmann odd vector, scalar and self-dual two-form respectively. The emergence of the Grassmann odd scalar also reflects the fact that that one of the supercharges has become a Lorentz scalar, thus opening up the possibility to define the twisted theory on an arbitrary curved manifold while preserving the scalar (topological) supersymmetry.

This procedure was generalized to the $N=4$ $d=4$ Yang-Mills theory by Yamron [16]. This is the dimensional reduction of $N=1$ $d=10$ Yang-Mills theory to $d=4$. As the $d=10$ spinor is Majorana-Weyl, there is no R-symmetry in $d=10$ and the R-symmetry group of the $d=4$ theory arises exclusively from the internal Lorentz group. Thus, decomposing the $N=1$ $d=10$ theory under $SO(10) \rightarrow SO(4) \times SO(6)$, one finds that the R-symmetry group is $SU(4)$.

Again, the twisting involves a choice of homomorphism from the Lorentz group to the global symmetry group, and the result of the twisting is most

usefully and compactly expressed in terms of how the (4) of $SU(4)$ decomposes under $SU(2)_L \times SU(2)_R$ [1]. There are three essentially different possibilities giving rise to theories with scalar supercharges, arising from the branchings

$$\begin{array}{rcl} (4) & \rightarrow & (2,1) \oplus (1,2) \\ (4) & \rightarrow & (1,2) \oplus (1,2) \\ (4) & \rightarrow & (1,2) \oplus (1,1) \oplus (1,1) \end{array} \tag{4.10}$$

respectively. As the fermions originally transform as

$$\text{Fermions:} \quad (2,1,4) \oplus (1,2,\bar{4}) \tag{4.11}$$

under $SU(2)_L \times SU(2)_R \times SU(4)$, descending from a $d\!=\!10$ Majorana-Weyl spinor with 16 real supercharges, in the twisted theory they transform as,

$$\begin{array}{rcl} \text{Fermions} & \rightarrow & 2(1,1) \oplus 2(2,2) \oplus (3,1) \oplus (1,3) \\ & \rightarrow & 2(1,1) \oplus 2(2,2) \oplus 2(1,3) \\ & \rightarrow & (1,1) \oplus (2,2) \oplus (1,3) \oplus 2(1,2) \oplus 2(2,1) \quad . \end{array} \tag{4.12}$$

This exhibits the third theory as the half-twisted model of [16], Donaldson-Witten theory coupled to spinorial (hypermultiplet) matter. The second theory, with a doubling of the Donaldson-Witten field content and $N_T\!=\!2$, is the A-model, i.e. the Euler character theory of instantons discussed in section 2 and constructed in this way by Yamron [16] and Vafa-Witten [1].

The first model, the B-model, is the one of interest here. Following [16, 1], the B-model was constructed by Marcus [13]. Its scalar field content follows from the observation that in the untwisted theory the scalars, as internal components of the $d\!=\!10$ connection, transform in the (6) of $SU(4) \sim SO(6)$ which is the antisymmetric product of the (4). Thus,

$$\text{Scalars} \rightarrow \wedge((2,1) \oplus (1,2)) = 2(1,1) \oplus (2,2) \tag{4.13}$$

and the bosonic field content consists of a connection, two scalars, and a one-form.

We observe that this is precisely the (reduced) field content of $d\!=\!4$ super-BF theory discussed above (after elimination of the auxiliary fields B and u), and in order to identify these two theories it remains to exhibit the $N_T\!=\!2$ symmetry in that model.

Spelling out the action (4.7) in detail one obtains

$$
\begin{aligned}
S_q^{(4),red} &= Q\int \chi F_A + d_AV*\chi + Q\int d_A\bar{\phi}*\psi - d_A\bar{\eta}*V \\
&= \int BF_A + \chi d_A\psi + *\chi d_A\bar{\psi} + [\psi,V]*\chi + d_AV*B \\
&+ \int -(d_A\eta*\psi + d_A\bar{\eta}*\bar{\psi}) - d_A\bar{\phi}*d_A\phi + d_Au*V \\
&+ \int [\psi,\bar{\phi}]*\psi - [\psi,\bar{\eta}]*V
\end{aligned} \tag{4.14}
$$

(all commutators are to be understood in the graded sense). One observes that there is almost a discrete symmetry exchanging ψ and $\bar{\psi}$ and η and $\bar{\eta}$ with $(u,V,\chi)\rightarrow(-u,-V,*\chi)$. To eliminate the 'almost', one can deform this action by adding to it

$$
\Delta S = Q\int -\tfrac{1}{2}[V,V]\chi + \tfrac{i}{2}\chi*B + [\bar{\psi},\bar{\phi}]*V \tag{4.15}
$$

(the factor of i being required in Euclidean space so that the B^2-term damps the amplitude $\exp iS$). Shifting $B\rightarrow B-id_AV$, one finds

$$
\begin{aligned}
S &= S_q^{(4),red} + \Delta S \\
&= \int B(F_A - \tfrac{1}{2}[V,V]) + \tfrac{i}{2}(B*B + d_AV*d_AV) \\
&+ \int d_Au*V - d_A\bar{\phi}*d_A\phi - \tfrac{i}{2}[\chi,*\chi]\phi \\
&+ \int (\chi d_A\psi + *\chi d_A\bar{\psi}) + ([\bar{\psi},\eta]*V - [\psi,\bar{\eta}]*V) - (d_A\eta*\psi + d_A\bar{\eta}*\bar{\psi}) \\
&+ \int ([\psi,V]*\chi - [\bar{\psi},V]\chi) - (\bar{\phi}[\bar{\psi},*\bar{\psi}] + \bar{\phi}[\psi,*\psi]) - [V,\phi]*[V,\bar{\phi}]
\end{aligned} \tag{4.16}
$$

This action is now manifestly invariant under

$$
(A,\psi,\bar{\psi},\phi,V,\bar{\phi},\eta,\bar{\eta},u,\chi,B)\rightarrow(A,\bar{\psi},\psi,\phi,-V,\bar{\phi},\bar{\eta},\eta,-u,*\chi,B) \tag{4.17}
$$

Consequently, in addition to the symmetry Q displayed in (4.4,4.6), the action has a symmetry $\bar{Q}$ following from combining Q with this discrete symmetry. Thus, e.g. the χ-transformation $Q\chi = B + id_AV$ (taking into account the shift of B) gives rise to the $\bar{Q}$-transformation $\bar{Q}\chi = *(B - id_AV)$.

The complete set of transformations is

$$\begin{aligned}
QA &= \psi & \bar{Q}A &= \bar{\psi} \\
Q\psi &= d_A\phi & \bar{Q}\psi &= -[V,\phi] \\
Q\phi &= 0 & \bar{Q}\phi &= 0 \\
Q\bar{\phi} &= \eta & \bar{Q}\bar{\phi} &= \bar{\eta} \\
Q\eta &= [\bar{\phi},\phi] & \bar{Q}\eta &= -u \\
QV &= \bar{\psi} & \bar{Q}V &= -\psi \\
Q\bar{\psi} &= [V,\phi] & \bar{Q}\bar{\psi} &= d_A\phi \\
Q\bar{\eta} &= u & \bar{Q}\bar{\eta} &= [\bar{\phi},\phi] \\
Qu &= [\bar{\eta},\phi] & \bar{Q}u &= -[\eta,\phi] \\
Q\chi &= B + id_AV & \bar{Q} &= *(B - id_AV) \\
QB &= [\chi,\phi] - i[\psi,V] + id_A\bar{\psi} & \bar{Q}B &= [*\chi,\phi] + i[\bar{\psi},V] + id_A\psi
\end{aligned} \tag{4.18}$$

The operators Q and $\bar{Q}$ are both equivariantly nilpotent, i.e. they square to gauge transformations generated by ϕ,

$$Q^2 = \bar{Q}^2 = \mathcal{L}_\phi \ . \tag{4.19}$$

They also anticommute, on-shell on χ and B and off-shell on all the other fields,

$$\begin{aligned}
\{Q,\bar{Q}\}\chi &= 2i\frac{\delta S}{\delta\chi} \\
\{Q,\bar{Q}\}B &= -2i[\frac{\delta S}{\delta B},\phi] \\
\{Q,\bar{Q}\} &= 0 \qquad \text{otherwise} \ .
\end{aligned} \tag{4.20}$$

To make contact with the action given by Marcus in [13], one can proceed to integrate out the field B. Then the first line of the action S becomes $(i/2)$ times

$$\int (F_A - \tfrac{1}{2}[V,V]) * (F_A - \tfrac{1}{2}[V,V]) + d_AV * d_AV = \int F_{A+iV} * F_{A-iV} \ . \tag{4.21}$$

This is precisely the bosonic part of the B-model action involving the complexified connections $A \pm iV$. To compare the other terms in the action, one notes that Marcus' fields, indicated by a subscript M in the following,

are related to those used here by complex linear combinations. With the dictionary

$$
\begin{aligned}
Q_M &= Q + i\bar{Q} \qquad & \bar{Q}_M &= Q - i\bar{Q} \\
\psi_M &= \bar{\psi} - i\psi \qquad & \chi_M &= -\tfrac{1}{2}[\chi + i * \chi] \\
\eta_M &= \bar{\eta} - i\eta \qquad & P_M &= 2u \\
C_M &= \tfrac{1}{2}\bar{\phi} \qquad & B_M &= 4\phi
\end{aligned}
\tag{4.22}
$$

one finds that the action and supersymmetry transformations given here match precisely those of [13] in the $\alpha \to 0$ limit. To obtain the action corresponding to $\alpha \neq 0$ one needs to add a term proportional to $u * u \sim -Q\bar{Q}\eta * \bar{\eta}$.

Thus we have shown that the topological B-twist of $N=4$ $d=4$ Yang-Mills theory studied in [13] can be regarded as a deformation of $d=4$ super-BF theory. At this particular point in the deformation space, the super-BF action exhibits an extended $N_T = 2$ supersymmetry. Conversely, if one is willing to sacrifice one of the supersymmetries, one can deform Marcus' action, which localizes onto complexified flat connections, to a theory which is manifestly real and localizes onto the moduli space of real flat connections (or, rather, taking into account the V zero modes, its tangent bundle). Note that this deformation, although it changes the moduli space, does not effect the topology of the moduli space one is localizing on. As such it is a legitimate, albeit somewhat unusual, deformation in the context of topological field theory.

In particular, formally observables and correlation functions in the two models agree, the additional $\bar{Q}$-symmetry in one of them just serving to pick out a particular representative of the Q-cohomology class. However, we will not dwell on this issue as it is not clear to us that these correlation functions are mathematically meaningful objects (for a discussion of some of the issues at stake see section 5.4.3 of [6] and section 5 of [13]). It is thus equally unclear if calculations in this model can be used as a test of S-duality in the spirit of [1].

4.3 A Novel Topological Twist of $N=4$ $d=3$ Yang-Mills Theory

The point of departure this time is the $N = 4$ $d = 3$ Yang-Mills theory, i.e. the dimensional reduction of the minimal $N=1$ $d=6$ or $N=2$ $d=4$

Yang-Mills theory to $d=3$. This theory has a global $SU(2)_E \times SU(2)_N \times SU(2)_R$ invariance (using the notation employed in [27]), where $SU(2)_E$ is the Euclidean group in $d=3$, $SU(2)_N$ is the internal Euclidean group arising from the decomposition $SO(6) \to SU(2)_E \times SU(2)_N$, and $SU(2)_R$ is the R-symmetry group of the six-dimensional theory (see [28] for other recent work on $N=4$ $d=3$ theories).

The field content of the untwisted dimensionally reduced model is a gauge field A and fermions and three scalars transforming as

$$\begin{array}{rl} \text{Fermions:} & (2,2,2) \\ \text{Scalars:} & (1,3,1) \end{array} \tag{4.23}$$

of $SU(2)_E \times SU(2)_N \times SU(2)_R$. There are two obvious ways of topologically twisting this model.

Twisting the $d=3$ Lorentz group $SU(2)_E$ by $SU(2)_R$ leads to the field content of the $d=3$ super-BF (or Casson, or Euler character, or super-IG) model [29, 17, 6, 7], namely a gauge field and twisted fermions and scalars transforming as

$$\begin{array}{rcl} \text{Fermions} & \to & (1,2) \oplus (3,2) \\ \text{Scalars} & \to & (1,3) \ , \end{array} \tag{4.24}$$

the second slot indicating the transformation behaviour under the remaining unbroken $SU(2)_N$. This is not surprising as this is just the dimensional reduction of Donaldson-Witten theory (the same twist being used in both cases). Interpreting the $SU(2)_N$ as a ghost-number symmetry, one can read off from the above that the model has an $N_T=2$ topological supersymmetry, as indeed we knew, the two charges Q and $\bar{Q}$ transforming as a doublet of $SU(2)_N$.

The second twist, by $SU(2)_N$, is intrinsically three-dimensional and leads to the field content

$$\begin{array}{rcl} \text{Fermions} & \to & (1,2) \oplus (3,2) \\ \text{Scalars} & \to & (3,1) \ . \end{array} \tag{4.25}$$

Consequently, as one has twisted by the internal Lorentz group and the scalars are the internal component of the $d=6$ gauge field, in this twisted theory the three scalars combine to form a $d=3$ one-form we will call V. Thus this theory has the somewhat unusal property of possessing no bosonic

scalars. The other fields are the gauge field A, Grassmann odd scalars η and $\bar{\eta}$, and Grassmann odd one-forms ψ and χ. Note that, once again, this theory will have an $N_T=2$ supersymmetry, the two supercharges Q and $\bar{Q}$ now transforming as a doublet of the R-symmetry group $SU(2)_R$.

As this twisted model differs from the first one by an exchange of $SU(2)_R$ and $SU(2)_N$, it is tempting to speculate that it can be regarded as providing a mirror description of the Casson invariant in the spirit of [28]. However, at present we have no solid evidence in favour of this.

One rather direct way of obtaining the action and supersymmetries of this model is to dimensionally reduce and twist the $N = 1$ $d = 6$ Euclidean Yang-Mills theory. The details of this procedure are straightforward but not particularly interesting, so we will only fill in some of the steps.

The Euclidean $N=1$ $d=6$ action is

$$L_E = -\Psi_R^\dagger \Gamma^M D_M \Psi_L - \tfrac{1}{4} F_{MN} F^{MN} \qquad (4.26)$$

where the fermions are chiral spinors in the (4) and $(\bar{4})$ of $Spin(6) = SU(4)$ and (in Euclidean space) the right-handed spinor $\Psi_R^\dagger$ is taken to be independent of Ψ_L. The Γ^M are Euclidean gamma matrices,

$$\{\Gamma^M, \Gamma^N\} = 2\delta^{MN} \quad . \qquad (4.27)$$

This action has the two independent supersymmetries

$$\begin{aligned} \delta\Psi_L &= \tfrac{1}{2}\Gamma^M \Gamma^N F_{MN} \epsilon_L \\ \delta\Psi_R^\dagger &= -\tfrac{1}{2}\epsilon_R^\dagger \Gamma^M \Gamma^N F_{MN} \\ \delta A_M &= -(\epsilon_R^\dagger \Gamma_M \Psi_L - \Psi_R^\dagger \Gamma_M \epsilon_L) \quad . \end{aligned} \qquad (4.28)$$

We will use a representation of the gamma matrices in $d=6$ which is well adapted to a $6 = 3 + 3$ decomposition. The basic building blocks are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \qquad (4.29)$$

satisfying

$$\begin{aligned} \sigma_k \sigma_l = i\epsilon_{klm}\sigma_m + \delta_{kl}\mathbb{I} \ , \\ \sigma_k^2 = \mathbb{I} \ , \qquad \sigma_k^\dagger = \sigma_k \ . \end{aligned} \qquad (4.30)$$

In terms of these, and the (2×2) identity matrix $\mathbb{I}$, one choice for the (8×8) gamma matrices is

$$\begin{aligned} \Gamma_k &= \sigma_1 \otimes \mathbb{I} \otimes \sigma_k \\ \Gamma_{k+3} \equiv \Gamma_a &= \sigma_2 \otimes \sigma_a \otimes \mathbb{I} \\ \Gamma_7 &= \sigma_3 \otimes \mathbb{I} \otimes \mathbb{I} \ . \end{aligned} \tag{4.31}$$

where the convention for tensor products is such that e.g. $\sigma_1 \otimes \mathbb{I}$ denotes the (4×4) matrix

$$\sigma_1 \otimes \mathbb{I} = \begin{pmatrix} 0_2 & \mathbb{I} \\ \mathbb{I} & 0_2 \end{pmatrix} \ . \tag{4.32}$$

Hence a $d=6$ Weyl spinor is of the form $\Psi_L = (\psi, 0)^t$, where ψ is a four-component complex spinor and an independent right-handed conjugate spinor is of the form $\Psi_R^\dagger = (0, \chi)$.

The anti-hermitian generators of $Spin(6)$ are thus

$$\begin{aligned} \tfrac{1}{2}[\Gamma_k, \Gamma_l] &= \mathbb{I} \otimes \mathbb{I} \otimes i\epsilon_{klm}\sigma_m \\ \tfrac{1}{2}[\Gamma_a, \Gamma_b] &= \mathbb{I} \otimes \epsilon_{abc}\sigma_c \otimes \mathbb{I} \\ \tfrac{1}{2}[\Gamma_k, \Gamma_a] &= i\sigma_3 \otimes \sigma_a \otimes \sigma_k \ . \end{aligned} \tag{4.33}$$

It follows from the above that under $SU(2)_N \times SU(2)_E \subset Spin(6)$, the chiral spinor ψ transforms as a $(2,2)$, with $SU(2)_E$, generated by $[\Gamma_k, \Gamma_l]/2$, acting diagonally on the two 3d spinors (i.e. the first two and last two components respectively), while they transform as a doublet under $SU(2)_N$. In other words, the action of $SU(2)_N \times SU(2)_E$ is generated by $\sigma_a \otimes \sigma_k$, and in terms of components $\psi_{A'B}$ one has

$$\psi_{A'B} \rightarrow \sigma_{aA'}{}^{C'} \sigma_{kB}{}^{D} \psi_{C'D} \ . \tag{4.34}$$

We will now take all fields to be independent of x^4, x^5, x^6. After twisting $SU(2)_E$ by $SU(2)_N$, the internal components A_c of the connection A transform as a covector V_k while the fermions transform as a scalar and a one-form,

$$\begin{aligned} \psi_{AB} &= \epsilon_{AB}\eta + \psi_k \sigma^k_{AB} \\ \chi^{AB} &= \epsilon^{AB}\bar{\eta} + \chi_k \sigma^{k\,AB} \ , \end{aligned} \tag{4.35}$$

where $\epsilon_{12} = \epsilon^{12} = 1$ and $\sigma^k_{AB} = \sigma^{k\,C}_{A}\epsilon_{CB}$ is symmetric.

The bosonic part of the Lagrangian becomes

$$\begin{aligned}
L_b &= -\tfrac{1}{4}F_{MN}F^{MN} \\
&\to -\tfrac{1}{4}(F_{ij}-[V_i,V_j])^2 - \tfrac{1}{4}(D_iV_j - D_jV_i)^2 - \tfrac{1}{2}(D_iV^i)^2 \\
&= -\tfrac{1}{4}F_{ij}(A+iV)F^{ij}(A-iV) - \tfrac{1}{2}(D_iV^i)^2 \ , \qquad (4.36)
\end{aligned}$$

and from the Dirac action one obtains

$$\begin{aligned}
L_f &= -\chi^{AB}\delta_A^C\sigma_{kB}{}^D D_k\psi_{CD} - i\chi^{AB}\sigma_{aA}{}^C\delta_B^D D_a\psi_{CD} \\
&\to 2\bar{\eta}D_k(A-iV)\psi_k - 2\chi_k D_k(A-iV)\eta \\
&\quad +2i\epsilon_{lkm}\chi_l D_k(A+iV)\psi_m \ . \qquad (4.37)
\end{aligned}$$

Hence, putting the two together one finds that the action of the twisted 3d model is

$$\begin{aligned}
S &= -\tfrac{1}{2}\int F_{A+iV} * F_{A-iV} + d_A * V * d_A * V \\
&\quad -2\int \bar{\eta}d_{A-iV} * \psi - \eta d_{A-iV} * \chi - i\chi d_{A+iV}\psi \ . \qquad (4.38)
\end{aligned}$$

To obtain the topological symmetries of this action, one needs to work out the scalar components $\epsilon_{L,AB} = \epsilon_L\epsilon_{AB}$ and $\epsilon_R^{\dagger AB} = \bar{\epsilon}\epsilon^{AB}$ of the supersymmetry transformations. Denoting the corresponding BRST operators by $\bar{Q}$ and Q respectively, one finds

$$\begin{aligned}
Q(A+iV) &= 4\psi & \bar{Q}(A+iV) &= -4\chi \\
Q(A-iV) &= 0 & \bar{Q}(A-iV) &= 0 \\
Q\psi &= 0 & \bar{Q}\psi &= -i * F_{A-iV} \\
Q\chi &= -i * F_{A-iV} & \bar{Q}\chi &= 0 \\
Q\eta &= 0 & \bar{Q}\eta &= -i * d_A * V \\
Q\bar{\eta} &= -i * d_A * V & \bar{Q}\bar{\eta} &= 0 \ . \qquad (4.39)
\end{aligned}$$

Note that the action has the discrete symmetry

$$(\psi,\chi,\eta,\bar{\eta}) \to (-\chi,\psi,-\bar{\eta},\eta) \qquad (4.40)$$

mapping Q to $\bar{Q}$, so that Q-invariance is equivalent to $\bar{Q}$ invariance. This symmetry can be interpreted as the action of the Weyl subgroup of the R-symmetry group $SU(2)_R$. This can be seen by introducing the $SU(2)$

doublets $i\mathbb{E}^A = (\bar{\eta}, \eta)$ and $\mathbb{P}^A = (\chi, \psi)$ in terms of which the fermionic Lagrangian can be written as

$$i\epsilon_{AB}(\mathbb{E}^A d_{A-iV} * \mathbb{P}^B - \frac{1}{2}\mathbb{P}^A d_{A+iV}\mathbb{P}^B) \ . \tag{4.41}$$

It is straightforward to see that indeed

$$QS = \bar{Q}S = 0 \ . \tag{4.42}$$

This can be made manifest by introducing some auxiliary fields which have the added virtue of making Q and $\bar{Q}$ nilpotent and anticommuting off-shell (so far they do so only modulo the η and $\bar{\eta}$ equations of motion). Thus, we introduce two auxiliary one-forms B and $\bar{B}$, and a scalar u with transformation rules (designed to preserve the above discrete invariance supplemented by $(B, \bar{B}, u) \to (B, \bar{B}, u)$)

$$\begin{aligned} Q\chi &= B & \bar{Q}\psi &= B \\ QB &= 0 & \bar{Q}B &= 0 \\ Q\bar{\eta} &= u & \bar{Q}\eta &= u \\ Qu &= 0 & \bar{Q}u &= 0 \\ Q\bar{B} &= -d_{A-iV}\eta & \bar{Q}\bar{B} &= d_{A-iV}\bar{\eta} \ . \end{aligned} \tag{4.43}$$

Then one has

$$Q^2 = \bar{Q}^2 = \{Q, \bar{Q}\} = 0 \tag{4.44}$$

off-shell. The action can now be written as a sum of a topological (BF like) term and a term which is actually $Q\bar{Q}$-exact,

$$S = 2i \int \bar{B}F_{A-iV} + \tfrac{i}{16}Q\bar{Q}S_{CS}(A+iV) - Q\bar{Q}\int iV * \bar{B} + \tfrac{1}{2}\eta * \bar{\eta} \ . \tag{4.45}$$

Here S_{CS} is the Chern-Simons action (with normalization $\int AdA + \ldots$), and the first term is Q and $\bar{Q}$ invariant by the Bianchi idenity. Integrating out $B, \bar{B}$ and u reproduces the action and transformation rules given above.

While we will leave a more detailed investigation of the localization properties and correlation functions in this model to future investigations, we do want to point out two rather striking features of this model which set it apart both from the B-model, discussed above, to which it bears a superficial resemblance, and from other known cohomological gauge theories.

One of the unusual features is that the supersymmetry is not equivariant in any sense (and indeed it hardly could be in the absence of the usual scalar ghost for ghost ϕ), but rather strictly nilpotent even prior to the introduction of gauge ghosts.

The second remarkable property is that $A - iV$ is both Q and $\bar{Q}$ invariant. Hence any gauge invariant functional of $A-iV$, constrained by $F_{A-iV} = 0$, is a candidate observable. In particular, here we have a cohomological theory in which bosonic Wilson loops appear to be good observables.

Finally, note that there is no net ghost number violation, as it should be since the unbroken R-symmetry group $SU(2)_R$ is simple. Hence the partition function and correlation functions of Wilson loops of $A - iV$ are potentially non-vanishing.

5 D-brane Instantons and Topological Gauge Theories in $d{=}2, 3$

In this section, we will discuss some other classes of low-dimensional topological gauge theories, arising from the dimensional reduction of $N{=}1$ $d{=}10$ or $N{=}4$ $d{=}4$ Yang-Mills theory to $d{=}3$ and $d{=}2$. As has been explained e.g. in [30], the dimensional reduction of $N{=}1$ $d{=}10$ super-Maxwell theory to $(p+1)$ dimensions describes the effective low-energy world-volume dynamics (i.e the collective coordinates) of flat Dirichlet p-branes (with enhanced $U(n)$ gauge symmetry for coincident D-branes).

Of course, the importance of the study of D-branes [31] for a deeper understanding of string theory need not be stressed and we just refer to two recent extensive reviews on D-branes [18] and string dualities [19] and the references therein for further information. What we will focus on in the following is a beautiful observation due to Bershadsky, Sadov and Vafa [3] that topologically twisted versions of these supersymmetric world-volume theories appear completely naturally in the study of curved D-branes and in particular for D-brane instantons wrapping around supersymmetric cycles [32] of the compactifying space. In particular, via an argument that we will recall below, they showed that all the three different twistings of $N{=}4$ $d = 4$ Yang-Mills theory exhibited in (4.12) appear in this way as effective world-volume theories, namely for special Lagrangian submanifolds of Calabi-Yau four-folds (the B-model [13]), for coassociative submanifolds of G_2-holonomy seven-manifolds (the A-model of [7] and Vafa-Witten [1]), and

for Cayley-submanifolds of Spin(7) eight-manifolds (the half-twisted model). Furthermore, the partial twist along a two-dimensional surface considered in [2] appears for three-branes wrapping a two-cycle of a K3.

With this in mind, we will in the following analyse the topological twistings of the $d=3$ theories. It turns out that there are only two (partial) topological twists (provided that one excludes theories involving higher spin fields) and that their field content and supersymmetries are precisely what one expects for the two known classes of supersymmetric three-cycles, namely special Lagrangian submanifolds of Calabi-Yau three-folds and associative submanifolds of G_2-holonomy manifolds. We regard it as rather pleasing (and intriguing) that also in this case all the topological twists can be naturally realized in this manner.

We will also briefly describe the topological gauge theories associated to Dirichlet one-brane (D-string) instantons wrapping holomorphic curves in K3s and Calabi-Yau three-folds.

Before proceeding to these lower dimensions, we would like to point out that twisted models can also be constructed under certain conditions in $d = 5$ and $d = 6$. As emntioned in the introduction, these appear to be related to considerations in [21] and [20] respectively. It should be interesting to study this further.

5.1 Topological Twists of $N=8$ $d=3$ Yang-Mills Theory

In this section, the theory of interest is once again the dimensional reductions of $N=1$ $d=10$ YM theory, this time to $d=3$.

Thus the global symmetry group is

$$H = SU(2)_E \times Spin(7) \quad , \tag{5.1}$$

under which the gauge field A, spinors and scalars transform as

$$\begin{aligned} \text{Gauge field:} &\quad (3,1) \\ \text{Fermions:} &\quad (2,8) \\ \text{Scalars:} &\quad (1,7) \quad , \end{aligned} \tag{5.2}$$

where (8) is the spinor representation of $Spin(7)$ and (7) is the vector representation.

Twists in $d=3$ involve decomposing the (8) of $Spin(7)$ under $SU(2)$. Clearly *a priori* there are many possibilities. However, the requirement that the spinor, which is a $(2,8)$, turn into something sensible in the twisted theory severely restricts the number of viable options.

First of all, in order that the twisted theory contain at least one scalar supercharge, a (2) of $SU(2)$ must occur in the decomposition of (8),

$$(8) \to (2) \oplus_i R_i \ \ . \tag{5.3}$$

Furthermore, among the R_i no representations of spin ≥ 1 should appear, because otherwise spin > 1 fermionic fields will appear in the twisted action. E.g. if one of the R_i were a (3) of spin one, from $(2)\otimes(3) = (2)\oplus(4)$ one would obtain a spin 3/2 field in the (4) of $SU(2)$. Hence, the second requirement is

$$\dim R_i \leq 2 \ \ . \tag{5.4}$$

Finally, for a full (as opposed to partial) topological twist, only half-integral spins should appear among the R_i so that the $(2) \otimes R_i$ are all tensorial.

A systematic search, using one's favourite reference on branchings (see e.g. [33]), reveals that there are essentially only two possibilities satisfying the first two desiderata, namely $(8) \to 4(2)$ and $(8) \to 2(2) \oplus 4(1)$, the former also satisfying the third and leading to a full topological twist. Both of them are most transparently described in terms of the branching

$$Spin(7) \to SU(2) \times SU(2) \times SU(2) \ \ , \tag{5.5}$$

which will make the maximal residual global symmetry group $SU(2)\times SU(2)$ manifest. Under (5.5), the (8) and (7) of $Spin(7)$ decompose as

$$\begin{aligned} (8) \quad &\to \quad (2,1,2) \oplus (1,2,2) \\ (7) \quad &\to \quad (2,2,1) \oplus (1,1,3) \ \ . \end{aligned} \tag{5.6}$$

Then, taking e.g. the diagonal of $SU(2)_E$ with the right-most $SU(2)$-factor one finds that in the twisted theory the spinors and scalars transform as

$$\begin{aligned} (2,8) \quad &\to \quad (1,2,1) \oplus (1,1,2) \oplus (3,2,1) \oplus (3,1,2) \\ (1,7) \quad &\to \quad (1,2,2) \oplus (3,1,1) \end{aligned} \tag{5.7}$$

under $SU(2)_E \times SU(2) \times SU(2)$. Therefore this twisted theory has $N_T=4$, the four scalar supercharges transforming as two $SU(2)$ doublets. The field

content consists of the $d\!=\!3$ gauge field A plus four Grassmann odd scalars η^A and $\bar{\eta}^{A'}$, four Grassmann odd vectors ψ^A and $\bar{\psi}^{A'}$ as well as four bosonic scalars $\phi^{AA'}$ and one bosonic one-form V (which is an $SO(4)$-singlet).

It is easy to see that this toplogical theory is precisely the dimensional reduction of either of the two $d\!=\!4$ $N_T\!=\!2$ theories. That both of these reduce to equivalent $d\!=\!3$ theories explains why there are only two topological twists of the $d\!=\!3$ theory whereas one might have expected at least three - arising from the dimensional reduction of the three topological twists of $N\!=\!4$ $d\!=\!4$ Yang-Mills theory.

The other topological twist is obtained by taking the diagonal of $SU(2)_E$ with the first of the three other $SU(2)$-factors. In this case one finds

$$\begin{aligned}(2,8) &\rightarrow (1,1,2)\oplus(3,1,2)\oplus(2,2,2)\\ (1,7) &\rightarrow (2,2,1)\oplus(1,1,3)\ \ .\end{aligned} \tag{5.8}$$

Thus this is a partially twisted $N_T = 2$ theory, and indeed precisely the dimensional reduction of the half-twisted $N_T = 1$ $d = 4$ theory. Its field content consists of the $d\!=\!3$ gauge field A plus two Grassmann odd scalars $\eta^{A'}$, two Grassmann odd vectors $\psi^{A'}$, four spinors $\lambda^{AA'}$, three scalars $\phi^{i'}$ (transforming as a vector under the second internal $SU(2)$) and two scalar spinors β^A.

This $N_T\!=\!2$ theory can for instance also be described in terms of the branching

$$Spin(7)\rightarrow G_2\rightarrow SU(2)\times SU(2)\ \ . \tag{5.9}$$

In that description, however, only the diagonal subgroup of the global symmetry group $SU(2)\times SU(2)$ is manifest.

The action of this theory can be described as the coupling of the $N_T = 2$ $d\!=\!3$ Euler character (super-BF, Casson, ...) theory to a hypermultiplet. Indeed the fields can be put into $N_T = 2$ superfields of section 3 augmented with $N_T = 2$ spinor superfields, thus making the topological invariance and symmetry manifest. We have not been able to find a superfield formulation for the $N_T = 4$ theory based on the superfields of section 3.6. However, a natural expectation is that for this theory (and indeed for all the topological theories that come from twisting reductions of the $N = 1$ $d = 10$ theory) one simply needs to twist the 10 dimensional superfields after dimensional reduction to obtain the topological superfields with manifest, but perhaps on-shell, supersymmetry.

Finally, we want to point out that one can construct precisely two other theories with $N_T > 0$ by twisting, both of which however involve fields transforming in the (4) of the twisted Lorentz group, i.e. spin-3/2 Rarita-Schwinger fields. An $N_T = 2$ theory of this kind can, for instance, be obtained by twisting once more the above $N_T = 4$ theory with one of the internal $SU(2)$s. The $N_T = 1$ theory follows e.g. from the chain of branchings

$$\begin{array}{lclclcl} Spin(7) & \to & SU(4) & \to & SU(3) & \to & SU(2) \\ (8) & \to & (4) \oplus (\bar{4}) & \to & (3) \oplus (\bar{3}) \oplus 2(1) & \to & (3) \oplus (2) \oplus 3(1) \\ (7) & \to & (6) + (1) & \to & (3) \oplus (\bar{3}) \oplus (1) & \to & (3) \oplus (2) \oplus 2(1) \end{array} \tag{5.10}$$

under which the spinors and scalars can be twisted to

$$\begin{array}{lcl} (2,8) & \to & (1) \oplus 4(2) \oplus (3) \oplus (4) \\ (1,7) & \to & 2(1) \oplus (2) \oplus (3) \quad . \end{array} \tag{5.11}$$

This observation may be of interest in its own right as consistent interacting higher spin theories are hard to come by even in flat space, while the twisting should make no difference there and thus map the consistent untwisted theory to another consistent theory. On the other hand it is certainly not guaranteed by the twisting procedure alone that these theories will make sense on a curved manifold.

5.2 Realizations as World-Volume Theories of 2-Brane Instantons

We now want to show that the two topological theories found above are naturally realised as world-volume theories of Dirichlet 2-brane instantons in type IIA string theory. For other work on Dirichlet p-brane instantons see e.g. [32, 34, 35, 36, 37].

The bosonic world volume fields of a (flat) Dirichlet p-brane are a world volume vector field A, arising from the boundary conformal field theory or the Chan-Paton factors of the open string theory, and $10 - (p + 1) = 9 - p$ scalar fields, the collective coordinates or Goldstone modes for the broken translation invariance. Now one can consider the situation where a curved Euclidean D-brane 'wraps' around a non-trivial cycle S of the compactifying manifold M (e.g. a Calabi-Yau manifold). For such a configuration to preserve some fraction of the supersymmetries left unbroken by the compactification, i.e. for the cycle to be supersymmetric [32], the cycle needs

to satisfy some rather stringent conditions identifying it [32, 3, 36, 37] as a calibrated submanifold [38, 39].

For later reference, let us collect here the relevant information concerning special-holonomy Ricci-flat manifolds. In the following table we have indicated the dimension m of the manifold, the holonomy group, the name usually given to such a manifold, the number of covariantly constant spinors (corresponding to the number N of supersymmetries in the compactified theory) and the fraction of supersymmetries thus preserved by a particular compactification.

$$\begin{array}{lllll} m=4 & SU(2) & \text{K3} & N=1 & 1/2 \\ m=6 & SU(3) & \text{Calabi-Yau} & N=1 & 1/4 \\ m=7 & G_2 & \text{Joyce} & N=1 & 1/8 \\ m=8 & SU(4) & \text{Calabi-Yau} & N=2 & 1/8 \\ m=8 & Spin(7) & \text{Joyce} & N=1 & 1/16 \end{array} \tag{5.12}$$

String compactifications on the first two types of manifolds are of course well known. Calabi-Yau four-folds appear in compactifications of F-theory [40, 41] and M-theory [42], and some aspects of compactifications of string theory and M-theory on Joyce manifolds have been studied in [43, 44, 37].

Let us now recall the argument of [3]. In general, even though a cycle may be supersymmetric it may not possess any covariantly constant spinors. Thus the supersymmetry of the world-volume theory cannot in general be realized in the standard form but will have to involve some twisted definition of the supercharges in order to give a meaning to the world-volume theory.

This comes about as follows: As the scalars are associated with translations of the D-brane, there should be only $10-\dim M \leq 9-p$ true scalars, while the remaining translational modes should organize themselves into a section of the normal bundle N_S to S in M. Thus these scalars are 'twisted' if the normal bundle is non-trivial, and so are then their superpartners, the fermions. The number N_T of scalar supercharges of the theory is obtained by matching $N_T/16$ with the fraction appearing in the last column of the above table. Thus, for a given supersymmetric cycle, knowledge of its normal bundle in, and the number of covariantly constant spinors on, the ambient manifold determine the bosonic field content and number of scalar supercharges of its $d=(p+1)$ dimensional world-volume dynamics. Conversely, given a (partially) topological gauge theory one can check if there are supersymmetric cycles with the requisite properties.

A slight refinement of these arguments also leads to information about some

global symmetries a topological gauge theory arising in this way should possess. Namely, thinking of it as arising from compactification on M, there should be a global invariance under the rotation group $SO(10-\dim M)$ (or $SO(9-\dim M,1)$) of the uncompactified dimensions. In particular, the true scalars should organize themselves into a vector in the fundamental representation of this group, with the other bosonic fields being singlets. This gives some *a priori* conditions on which branchings of the R-symmetry group $SO(9-p)$ can lead to topological theories associated with some manifold M - those that proceed via the branching

$$SO(9-p) \to SO(10-\dim M) \times SO(\dim M - p - 1) \tag{5.13}$$

and subsequent twisting by the second factor (so that the first factor is preserved and only the normal directions to S in M are affected by the twisting). This is indeed what we will find below.

As an example, consider the B-model of section 4.2. Its bosonic field content consists of the world-volume gauge field as well as of two true scalars and one four-vector V. The fact that $d=(p+1)=4$ and that there are two scalars together imply that one is looking for a four-cycle S in an eight-dimensional manifold M. $N_T=2$ indicates that one is looking for an eight-manifold preserving $2/16=1/8$ of the supersymmetry. Thus M is a Calabi-Yau four-fold and S is a special Lagrangian submanifold, i.e. a submanifold for which the real part of the holomorphic four-form restricts to the volume form. A consistency check is provided by the fact that the normal bundle N_S can in that case indeed be identified with the cotangent bundle [39], accounting for the fact that the remaining four scalars make up a one-form V on S. Finally, we note that this theory has a global $U(1)$ ghost-number symmetry, and that - in accordance with the observation in the previous paragraph - this global symmetry can be identified with the rotation group $SO(2)$ in the two uncompactified dimensions, the two true scalars having $U(1)$-charges $(2,-2)$ and hence comprising a two-dimensional vector. In a similar fashion, the other two topological twists of $N=4$ $d=4$ Yang-Mills can be identified with world-volume theories of D-brane instantons [3] leading to the result recalled at the beginning of this section.

With this in mind, let us now turn to the two three-dimensional topological theories obtained above. In the $N_T=4$ theory, instead of the seven true scalars of the untwisted theory we have only four true scalars and one one-form V. Thus we are looking for a six-manifold M which preserves $4/16=1/4$ of the supersymmetries, i.e. a Calabi-Yau three-fold. As in the $d=4$

case above, the normal bundle of a special Lagrangian submanifold S in M can be identified with the contangent bundle T^*_S, in agreement with the appearance of V as the remaining bosonic field. Note that indeed, as anticipated above, the internal $SU(2) \times SU(2)$ can be thought of as the $SO(4)$ of the uncompactified dimensions, the corresponding four scalars transforming as a vector of $SO(4)$ while V is an $SO(4)$-singlet,

In the $N_T = 2$ theory, on the other hand, we have three true scalars transforming as a vector of one internal $SU(2)$ and two spinors β^A transforming as a doublet of the other. By the now familiar argument we expect to be dealing with a seven-manifold possessing one covariantly constant spinor (i.e. preserving 1/8 of the supersymmetries). Thus M is a Joyce [45] seven-manifold of G_2-holonomy.

It is known [39] that an associative $d=3$ submanifold S of a G_2 7-manifold has a normal bundle $N = \mathbb{S} \otimes V$, where $\mathbb{S}$ is the spinor bundle of S and V is a rank two $SU(2)$-bundle. But this fits in perfectly with the fact that the twisted scalars β^A are an $SU(2)$-doublet of spinors on S. Note also that once again the three true scalars organize themselves into a vector of the orthogonal group $SO(3)$ of the uncompactified dimensions.

Thus we have succesfully identified the two topological twists of $N = 4$ $d = 3$ Yang-Mills theory with the world-volume theories of Dirichlet two-brane instantons wrapped around the two known classes of supersymmetric three-cycles.

That these identifications are correct can also be deduced from the dimensional reduction of the results of [3] using the local models (coordinate representations) of calibrated manifolds given e.g. by McLean [39].

Let us start with the half-twisted $N_T = 2$ model. Recall that, according to [3], the $d=4$ half-twisted model corresponds to a so-called Cayley submanifold of a $Spin(7)$ 8-manifold, characterized by a Cayley 4-form. It is easy to see that integration of this form over a toroidal fibre produces the three-form characterizing associative submanifolds of G_2-holonomy 7-manifolds, in agreement with the above identification.

Concretely this means the following: Let the Cayley 4-form in local coordinates (or globally on the frame bundle) be given by ([0123] stands for $dx^0 dx^1 dx^2 dx^3$ etc. and the product is the wedge product)

$$\begin{aligned}\Omega_{Cayley} &= [0123] + ([01] - [23])([45] - [67]) + ([02] - [13])([46] + [57]) \\ &\quad + ([03] - [12])([47] - [56]) + [4567] \qquad (5.14)\end{aligned}$$

(see ([39, eq. (6.1)]). Now integrate over the fibre x^0 to obtain

$$\pi_* \Omega_{Cayley} = [123] + [1]([45] - [67]) + [2]([46] + [57]) + [3]([47] - [56]) \quad . \tag{5.15}$$

According to [39, eq. (5.1)] this is precisely the three-form Ω_{ass} characterizing associative 3-manifolds of G_2-manifolds,

$$\pi_* \Omega_{Cayley} = \Omega_{ass} \quad . \tag{5.16}$$

For the $N_T\!=\!4$ model there are two different 4-dimensional origins, as both $N_T\!=\!2$ $d\!=\!4$ theories reduce to it in $d\!=\!3$. On the one hand, according to [3], the Euler character theory of [7] and [1], i.e. the A-model, corresponds to coassociative submanifolds of G_2-manifolds, characterized by the Hodge dual 4-form $\Omega_{coass} = *\Omega_{ass}$ which, by a relabelling, we will write as (cf. [39, eq. (4.4)])

$$\Omega_{coass} = [0123] - [56]([01] - [23]) + [46]([02] + [13]) - [45]([03] - [12]) \quad . \tag{5.17}$$

Integrating this over x^0, one obtains

$$\pi_* \Omega_{coass} = [123] - [156] + [246] - [345] \quad . \tag{5.18}$$

This ought to be compared with the (real part of the) holomorphic volume form of a Calabi-Yau 3-fold which characterizes special Lagrangian submanifolds. The local model for this is the following. Let $z^k = x^k + ix^{k+3}$ be local complex coordinates. Then

$$\begin{aligned} \Omega_{CY} &= dz^1 \wedge dz^2 \wedge dz^3 \qquad (5.19) \\ &= ([123] - [453] - [156] - [426]) + i([423] + [513] + [612] - [456]) \quad , \end{aligned}$$

so that indeed

$$\pi_* \Omega_{coass} = \mathrm{Re}\Omega_{CY} \quad , \tag{5.20}$$

as expected.

On the other hand, according to [3], the B-model is realized on special Lagrangian submanifolds of Calabi-Yau 4-folds. This situation is slightly different as now the dimension of the ambient space changes by two in comparison with special Lagrangian submanifolds of Calabi-Yau 3-folds featuring in the dimensionally reduced theory. But if one assumes that the 4-fold is locally of the form $CY_4 = CY_3 \times T^2$ (i.e. it is an elliptic fibration), then clearly special Lagrangian submanifolds of CY_4 wrapping around one of the circles reduce (upon double dimensional reduction) to special Lagrangian submanifolds of CY_3, showing again the consistency of the dimensional reduction procedure with the geometrical interpretation in terms of calibrated submanifolds.

5.3 Topological Gauge Theories on Holomorphic Curves

In a similar spirit one can now discuss the theories associated with Dirichlet one-brane (or D-string) instantons. Ignoring the world-volume gauge field (which has no local dynamics) one would, by $SL(2,\mathbb{Z})$ duality of the type IIB string, expect D-string instantons to correspond to standard world-sheet instantons. And indeed the only known supersymmetric two-cycles are holomorphic curves in Calabi-Yau n-folds. In the following we will focus on the two cases $n = 2$ and $n = 3$ and briefly illustrate how the various facets of the corresponding world-volume theories we have encountered for $d = 3$ above fall into place here.

The starting point is again the dimensional reduction of $N=1\, d=10$ Yang-Mills, this time to $d=2$. The resulting theory has $N=8$ supersymmetry, and the R-symmetry group is now $SO(8)$ or $Spin(8)$. The field content consists of a $d = 2$ gauge field A and the fermions and scalars (the transverse fluctuations) which transform as

$$\begin{array}{rcl} \text{Fermions} & \rightarrow & 8_c^{+1} \oplus 8_s^{-1} \\ \text{Scalars} & \rightarrow & 8_v^{0} \end{array} \tag{5.21}$$

under $Spin(8) \times U(1)$. Another way of saying this is that left-movers live in the 8_c and right-movers in the 8_s. Modulo the (locally trivial) gauge field, these are precisely the same fluctuations as those of the fundamental IIB string, a manifestation of the self-duality of the IIB string.

Clearly, two (partial) topological twists of the $N=8$ $d=2$ theory can be obtained as dimensional reductions of the $N_T=2,4$ $d=3$ theories discussed in the previous section. As it turns out, these are precisely the two theories we are after which describe the situations of interest, namely holomorphic curves in K3s and Calabi-Yau three-folds.

Let us start with the $N_T=4$ $d=3$ theory. It follows from the description of the theory given after (5.7) that the dimensionally reduced field content is given by the gauge field, 8 Grassmann odd scalars (hence this theory has $N_T=8$), 4 Grassmann odd one-forms, six bosonic scalars and one bosonic one-form. Exhibiting their quantum numbers under the symmetry group $SU(2) \times SU(2)$ manifest in (5.7) (we will see later that this theory actually has a larger $SU(4)$ global symmetry), we see that this topological twist amounts to

$$8_c^{+1} \oplus 8_s^{-1} \quad \rightarrow \quad 2(2,1)^0 \oplus 2(1,2)^0 \oplus (2,1)^{+2,-2} \oplus (1,2)^{+2,-2}$$

$$8_v^0 \quad \rightarrow \quad (2,2)^0 \oplus 2(1,1)^0 \oplus (1,1)^{+2,-2} \quad . \tag{5.22}$$

Here the two $SU(2) \times SU(2)$ singlet scalars correspond to the internal components A_3 and V_3 of the $d = 3$ gauge field and one-form.

In the same way, the $N_T = 2$ theory (5.8) reduces to an $N_T = 4$ theory with bosonic field content a $d=2$ gauge field, 4 scalars and two spinors, the twisting being described by

$$\begin{aligned} 8_c^{+1} \oplus 8_s^{-1} \quad &\rightarrow \quad 2(1,2)^0 \oplus (1,2)^{+2,-2} \oplus (2,2)^{+1,-1} \\ 8_v^0 \quad &\rightarrow \quad (2,1)^{+1,-1} \oplus (1,3)^0 \oplus (1,1)^0 \quad . \end{aligned} \tag{5.23}$$

To find the geometrical interpretation or realization of these theories, we need to know something about supersymmetric two-cycles. As mentioned above, the only known supersymmetric 2-cycles are holomorphic curves in Calabi-Yau n-folds. One might think that another candidate are special Lagrangian submanifolds of a K3 (Calabi-Yau 2-fold). Indeed, at first sight a special Lagrangian submanifold (in particular, the Kähler form restricts to zero) and a holomorphic curve (the Kähler form restricts to the volume form) appear to be very different. However, K3's are actually hyper-Kähler so that there exists a triplet I, J, K of complex structures and it turns out [46] that a curve is special Lagrangian with respect to I iff it is holomorphic with respect to K, so that indeed the special Lagrangian case need not be considered seperately.

As before, identifications of the twisted world-sheet theories require considerations of the normal bundle. A preliminary consideration is the following: the manifolds we are dealing with are Calabi-Yau n-folds M_n. Hence they satisfy $c_1(T_M) = 0$ where T_M is the holomorphic tangent bundle. Thus, for a holomorphic genus g curve S in M_n one has

$$\begin{aligned} &T_M|_S = T_S \oplus N_S \\ &c_1(N_S) = -c_1(T_S) = 2g - 2 \quad . \end{aligned} \tag{5.24}$$

Alternatively, one notes that $\wedge^n T_M$ is trivial, so that one has

$$1 = \wedge^n T_M|_S = T_S \wedge^{n-1} N_S \tag{5.25}$$

which gives the constraint

$$\wedge^{n-1} N_S = K_S \quad , \tag{5.26}$$

K_S the canoncial line bundle on S (as $T_S = K_S^{-1}$).

Let us quickly dispose of the trivial case $n = 1$. Then M is just a torus T^2, so that what we are looking at is 'compactification' on T^2 and a D-string whose world-sheet is that T^2. The normal bundle is trivial, so the scalar field content should be eight scalars. In addition, no further supersymmetries are broken by this compactification, so the world-sheet theory is just Euclidean $N=8$ $d=2$ SYM with $N_T=16$ real supercharges (switching freely between supersymmetric and topological notation as it makes no difference on T^2).

When $n = 2$, we are dealing with holomorphic curves in a K3. This case has essentially been studied from this point of view in [3], cf. also [47]. N_S has complex rank one and the condition (5.26) is solved uniquely by $N_S = K_S$. Compactification on K3 leaves 6 flat directions so we expect the scalar field content to be 6 scalars and a one-form. This is just the field content of the $N_T = 8$ model (the dimensional reduction of the fully topological $N_T = 4$ $d=3$ theory). As K3 breaks half the supersymmetries, this matches with $N_T = 8$.

However, according to the general argument put forward in the previous section, the six scalars should transform as a vector of $SO(6)$ which should be a global symmetry group of this theory - whereas only $SU(2) \times SU(2)$ is manifest in (5.22). To exhibit this $SO(6) \sim SU(4)$ symmetry of the theory, we proceed as follows. We consider the branching (cf. (5.13))

$$Spin(8) \rightarrow SU(4) \times U(1) \quad , \tag{5.27}$$

under which the representations $8_{v,s,c}$ decompose as

$$\begin{array}{rcl} 8_v & \rightarrow & (1)^{+2,-2} \oplus (6)^0 \\ 8_s & \rightarrow & (4)^{+1} \oplus (\bar{4})^{-1} \\ 8_c & \rightarrow & (4)^{-1} \oplus (\bar{4})^{+1} \quad . \end{array} \tag{5.28}$$

Thus, twisting the Lorentz $U(1)$-charge by the internal $U(1)$-charge (by simply adding them up) and leaving the $SU(4)$ intact, one finds that the field content of the twisted theory is

$$\begin{array}{rcl} 8_c^{+1} \oplus 8_s^{-1} & \rightarrow & 2(4)^0 \oplus (\bar{4})^{+2,-2} \\ 8_v^0 & \rightarrow & (6)^0 \oplus (1)^{+2,-2} \quad . \end{array} \tag{5.29}$$

This has the expected manifest global $SU(4)$ symmetry under which the six scalars transform as a vector, and it reduces to the field content with the

quantum numbers as in (5.22) under the further branching

$$SU(4) \to SU(2) \times SU(2) \times U(1) \ . \tag{5.30}$$

When $n = 3$, we are dealing with holomorphic curves in Calabi-Yau 3-folds. This has been thoroughly studied (for a recent review see [48]), and our brief presentation of the results will certainly not do justice to the complexity and diversity of the subject.

In this case, the condition (5.26) on N_S reads $\wedge^2 N_S = K_S$ and generally this is solved by

$$N_S = K_S^{1/2} \otimes V, \tag{5.31}$$

where V is a rank two bundle with trivial determinant. This includes as a special case unobstructed rational curves for which one has

$$T_M|_S = \mathcal{O}_S(2) \oplus \mathcal{O}_S(-1) \oplus \mathcal{O}_S(-1) \ , \tag{5.32}$$

and for which V is trivial, as well as other situations such as $V = K_S^{1/2} \oplus K_S^{-1/2}$, which could appear for $M_3 = K3 \times T^2$ (we will briefly come back to that case below). Thus the bosonic field content of the world-sheet theory of the D-string instanton wrapping around such a holomorphic curve should consist of 4 scalars (for the uncompactified directions) and a doublet of spinors. This is exactly the field content of the $N_T\!=\!4$ theory (5.23), the dimensional reduction of the half-twisted $N_T\!=\!2$ $d\!=\!3$ theory, and again the supersymmetries work out as a Calabi-Yau 3-fold breaks 1/4 of the supersymmetries, leaving one with $N_T = 16/4 = 4$. Geometrical considerations, as in (5.13), suggest that this twisted theory should arise from the branching $SO(8) \to SO(4) \times SO(4)$ or

$$Spin(8) \to SU(2) \times SU(2) \times SU(2) \times SU(2) \ , \tag{5.33}$$

and it can indeed be seen that the corresponding branchings

$$\begin{aligned} 8_v &\to (2,2,1,1) \oplus (1,1,2,2) \\ 8_s &\to (1,2,1,2) \oplus (2,1,2,1) \\ 8_c &\to (1,2,2,1) \oplus (2,1,1,2) \end{aligned} \tag{5.34}$$

reproduce the field content (5.23) with an extended $SO(4) \times SU(2)$ global symmetry corresponding to rotations in the uncompactified dimensions and on the internal $SU(2)$ associated with the vector bundle V.

Let us now come back to the special case where the Calabi-Yau three-fold is of the form $K3\times T^2$ with holonomy $SU(2)\subset SU(3)$. In that case, one would expect an enlarged topological symmetry to be present. This is indeed the case and comes about as follows.

If e.g. $S\subset K3$, then its normal bundle in $K3$ can be identified with K_S and thus its normal bundle in $K3\times T^2$ is

$$N_S = K_S \oplus \mathcal{O}_S \quad . \tag{5.35}$$

This is of the form (5.31) for $V = K_S^{1/2}\oplus K_S^{-1/2}$. Thus the structure group of V has been reduced to $U(1)$. The identification $K_S^{1/2}\otimes V \sim K_S\oplus\mathcal{O}_S$ is then realized on the fields by a further twisting by this $U(1)$. As usual, this turns the $SU(2)$ spinor doublet into a one-form and two scalars. Its effect on the Grassmann odd field content displayed in the first line of (5.23) is to turn it into eight Grassmann odd scalars and four one-forms. Thus this twisted model shows the expected increase in the number of topological symmetries from $N_T=4$ to $N_T=8$. In fact, not too surprisingly, this model coincides with the $N_T=8$ model (5.22), the difference between $\mathbb{R}^2$ and T^2 (i.e. the fact that the deformations are compact for the latter) being invisible in the low-energy effective action. Similar remarks apply to other situations in which the holonomy group of the n-fold is a strict subgroup of $SU(n)$.

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